

One proof of the original Kemer's theorems
(concerning the text of C. Procesi "What happened to PI-theory"
arxiv.org/abs/1403.5673).

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March 07, 2015.

Abstract

We consider associative algebras over a field of characteristic zero. We give a version of the proof of the Kemer's theorems concerning the Specht problem solution [12], [13]-[16]. It is proved that the ideal of graded identities of a finitely generated PI-superalgebra coincides with the ideal of graded identities of some finite dimensional superalgebra. This implies that the ideal of polynomial identities of any (not necessary finitely generated) PI-algebra coincides with the ideal of identities of the Grassmann envelope of a finite dimensional superalgebra, and is finitely generated as a T-ideal.

MSC: Primary 16R50; Secondary 16R10, 16W50, 16W22

Keywords: Associative algebras, superalgebras, graded identities, PI-algebras.

Introduction

We present here one of the proofs of the Kemer's theorems about Specht problem solution for associative PI-algebras over a field of characteristic zero [12]. We give principally the proof of the fact that any finitely generated $\mathbb{Z}/2\mathbb{Z}$ -graded PI-algebra has the same graded identities as some finite dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. This result is the crucial and the most difficult step of the whole Kemer's solution of the Specht problem. The proof represented here is the partial case of the proof of a similar result for algebras graded by a finite abelian group given by the author in [23] (the case $G = \mathbb{Z}/2\mathbb{Z}$). It is only slightly modified to extend the result from an algebraically closed base field (as it was considered in [23]) to any field of characteristic zero. Observe that the proof of generalised results in [23] is independent on the original Kemer's proof. The unique essential reference to the Kemer's results

in that paper served to obtain Lemma 1. This reference can be safely exchanged by the Lewin's results [17]. PI-representability of a non-graded finitely generated PI-algebra can be also obtained as a partial case of [23] considering the trivial group $G = \{e\}$.

The second step of the Kemer's arguments is to prove that any PI-algebra satisfies the same graded polynomial identities as the Grassmann envelope of some finitely generated $\mathbb{Z}/2\mathbb{Z}$ -graded PI-algebra. We refer the reader to the book [9] for the proof of this fact. The author also can prove this fact independently but assume that the proof represented in [9] is the most elegant, short and clear to understand for a reader. Thus the author thinks that there is no any need to repeat these arguments. These two theorems imply (see also [9]) that any PI-algebra satisfies the same polynomial (non-graded) identities as the Grassmann envelope of some finite dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. The positive solution of the Specht problem in the classical case follows from the last classification theorem immediately.

Observe that besides the original proof of Aleksandr Kemer [13]-[16], published later also in [12], there are several later versions of various authors including interpretations and generalisations. Therefore the author does not pretend on originality or some exclusive properties of this text. And she never would get an idea to rewrite the proof of the original Kemer's results whenever some obvious misunderstandings appeared some time ago. This text is just one of the possible version of the proof as the author understood it generalising these results in [23]. Notice that this text is a little more detailed then the text published in [23]. Even here we omit some details and proofs that seems to us not very difficult to restore for a reader. But the author can assure possible readers that all the statements with omitted arguments were really checked. Moreover, even the text [23] is the restricted version of the original detailed author's text.

Throughout the paper we consider only associative algebras over a field of characteristic zero (not necessary unitary). Further they will be called algebras.

Let F be a field of characteristic zero and $F\langle X \rangle$ the free associative algebra over F generated by a countable set $X = \{x_1, x_2, \dots\}$. A T-ideal of $F\langle X \rangle$ is a bilateral ideal invariant under all endomorphisms of $F\langle X \rangle$.

Let A be an associative algebra over F . A polynomial $f = f(x_1, \dots, x_n) \in F\langle X \rangle$ is called polynomial identity for A if $f(a_1, \dots, a_n) = 0$ for any $a_1, \dots, a_n \in A$ ($f \equiv 0$ in A). Let us denote by $\text{Id}(A) = \{f \in F\langle X \rangle \mid f \equiv 0 \text{ in } A\}$ the ideal of all polynomial identities of A . If A satisfies a non-trivial polynomial identity then A is called *PI*-algebra. It is well known, for example, that any finite dimensional algebra is PI ([10, 18, 21]). The relation between T-ideals of $F\langle X \rangle$ and *PI*-algebras is well understood: for any F -algebra A , $\text{Id}(A)$ is a T-ideal of $F\langle X \rangle$, and any T-ideal I of $F\langle X \rangle$ is the ideal of identities of some F -algebra A , in particular, of the relatively free algebra of $F\langle X \rangle/I$.

An algebra A is called $\mathbb{Z}/2\mathbb{Z}$ -graded (or super-algebra) if $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is the direct sum of its subspaces $A_{\bar{0}}, A_{\bar{1}}$ where $A_{\theta}A_{\xi} \subseteq A_{\theta\xi}$ holds for any $\theta, \xi \in \mathbb{Z}/2\mathbb{Z}$. An element $a \in A_{\theta}$ is called homogeneous in the $\mathbb{Z}/2\mathbb{Z}$ -grading of the graded degree θ , we also write $\deg_{\mathbb{Z}_2} a = \theta$ in this case. The homogeneous component $A_{\bar{0}}$ of the

$\mathbb{Z}/2\mathbb{Z}$ -graded algebra A is called neutral (or even).

A homomorphism of $\mathbb{Z}/2\mathbb{Z}$ -graded algebras A, B $\varphi : A \rightarrow B$ is graded if $\varphi(A_\theta) \subseteq B_\theta$ for any $\theta \in \mathbb{Z}/2\mathbb{Z}$. An ideal $I \trianglelefteq A$ of a graded algebra A is graded if and only if I is generated by homogeneous in the grading elements. In this case the quotient algebra A/I is also $\mathbb{Z}/2\mathbb{Z}$ -graded with the grading induced by the grading of A ($\deg_\theta \bar{a} = \deg_\theta a$).

Let us denote by $A_1 \times \cdots \times A_\rho$ the direct product of algebras A_1, \dots, A_ρ , and by $A_1 \oplus \cdots \oplus A_\rho \subseteq A$ the direct sum of subspaces A_i of an algebra A . We also denote by $J(A)$ the Jacobson radical of A . Observe that in general all bases and dimensions of spaces and algebras are defined over the base field F unless otherwise indicated.

We always assume that the set \mathbb{N}_0^k ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) is linearly ordered with the lexicographical order for any natural k .

The most part of notions, definitions, facts and properties of polynomial identities may be found in [5], [7], [8], [9], [12].

1 Free graded algebra.

Let us denote by $X^{\mathbb{Z}_2} = \{x_{i\theta} | i \in \mathbb{N}, \theta \in \mathbb{Z}/2\mathbb{Z}\}$ a countable set of pairwise different elements. The algebra $\mathfrak{F} = F\langle X^{\mathbb{Z}_2} \rangle$ is the free associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra with the grading $\mathfrak{F} = \bigoplus_{\theta \in \mathbb{Z}/2\mathbb{Z}} \mathfrak{F}_\theta$, where $\mathfrak{F}_\theta = \langle x_{i_1\theta_1} x_{i_2\theta_2} \cdots x_{i_s\theta_s} | \theta = \theta_1 + \theta_2 + \cdots + \theta_s \rangle_F$. Since all the variables are pairwise distinct then any multihomogeneous polynomial is also a homogeneous element of \mathfrak{F} in the sense of the grading. An element $x_{i\theta}$ is called graded variables, and $f \in \mathfrak{F}$ a graded polynomial.

Let $f = f(x_{1\theta_1}, \dots, x_{n\theta_n}) \in F\langle X^{\mathbb{Z}_2} \rangle$ be a non-trivial $\mathbb{Z}/2\mathbb{Z}$ -graded polynomial. We say that a $\mathbb{Z}/2\mathbb{Z}$ -graded algebra A satisfies the graded identity $f(x_{1\theta_1}, \dots, x_{n\theta_n}) = 0$ if $f(a_{1\theta_1}, \dots, a_{n\theta_n}) = 0$ in A for any $a_{i\theta_i} \in A_{\theta_i}$. Denote by $\text{Id}^{\mathbb{Z}_2}(A) \trianglelefteq F\langle X^{\mathbb{Z}_2} \rangle$ the ideal of $\mathbb{Z}/2\mathbb{Z}$ -graded identities of A . Similar to the case of ordinary (non-graded) identities $\text{Id}^{\mathbb{Z}_2}(A)$ is a two-side graded ideal of $F\langle X^{\mathbb{Z}_2} \rangle$ invariant under all graded endomorphisms of $F\langle X^{\mathbb{Z}_2} \rangle$. Such ideals are called \mathbb{Z}_2 T-ideals. It is clear that any \mathbb{Z}_2 T-ideal I of $F\langle X^{\mathbb{Z}_2} \rangle$ is the ideal of $\mathbb{Z}/2\mathbb{Z}$ -graded identities of $F\langle X^{\mathbb{Z}_2} \rangle / I$.

Take a set $S \subseteq F\langle X^{\mathbb{Z}_2} \rangle$. Denote by $\mathbb{Z}_2T[S]$ the \mathbb{Z}_2 T-ideal generated by S . Then $\mathbb{Z}_2T[S]$ contains exactly all graded identities which are consequences of polynomials of the set S . $\mathbb{Z}/2\mathbb{Z}$ -graded algebras A and B are called \mathbb{Z}_2 PI-equivalent ($A \sim_{\mathbb{Z}_2} B$) if $\text{Id}^{\mathbb{Z}_2}(A) = \text{Id}^{\mathbb{Z}_2}(B)$. Let Γ be a \mathbb{Z}_2 T-ideal. We write also $f = g \pmod{\Gamma}$ for $f, g \in F\langle X^{\mathbb{Z}_2} \rangle$ iff $f - g \in \Gamma$.

We consider only graded identities of PI-superalgebras, i.e. the case when $\text{Id}^{\mathbb{Z}_2}(A) \supseteq \Gamma$ for some nonzero ordinary T-ideal $\Gamma \trianglelefteq F\langle X \rangle$. This holds iff the neutral component A_e is a PI-algebra ($\text{Id}^{\mathbb{Z}_2}(A) \ni f(x_{1\bar{0}}, \dots, x_{n\bar{0}}) \neq 0$) (see [6], and [1]).

Observe that a finitely generated superalgebra always has a finite set of $\mathbb{Z}/2\mathbb{Z}$ -homogeneous generators. We consider only homogeneous generating sets of superalgebras.

Definition 1 *Given a finitely generated superalgebra A and a finite homogeneous generating set K of A denote by $\text{grk}(K)$ the maximal number of elements of the*

same graded degree in K .

Then the homogeneous rank $\text{grk}(A)$ of A is the least $\text{grk}(K)$ for all finite homogeneous generating sets K of A .

Let $X_\nu^{\mathbb{Z}_2} = \{x_{i\theta} | 1 \leq i \leq \nu; \theta \in \mathbb{Z}/2\mathbb{Z}\}$ be a finite set of graded variables and $F\langle X_\nu^{\mathbb{Z}_2} \rangle$ the free associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra of the rank ν , $\nu \in \mathbb{N}$.

Let A be a finitely generated $\mathbb{Z}/2\mathbb{Z}$ -graded algebra. Then the superalgebra $U_\nu = F\langle X_\nu^{\mathbb{Z}_2} \rangle / (\text{Id}^{\mathbb{Z}_2}(A) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle)$ is the relatively free algebra of the rank ν for A . Similarly to case of ordinary identities if $\nu \geq \text{grk}(A)$ then $\text{Id}^{\mathbb{Z}_2}(A) = \text{Id}^{\mathbb{Z}_2}(U_\nu)$. Moreover the next remark takes place.

Given a \mathbb{Z}_2 T-ideal $\Gamma_1 \subseteq F\langle X_\nu^{\mathbb{Z}_2} \rangle$ denote by $\Gamma_1(F\langle X_\nu^{\mathbb{Z}_2} \rangle) = \{f(h_1, \dots, h_n) | f \in \Gamma_1, h_i \in F\langle X_\nu^{\mathbb{Z}_2} \rangle, \deg_{\mathbb{Z}_2} h_i = \deg_{\mathbb{Z}_2} x_i, \forall i\} \trianglelefteq F\langle X_\nu^{\mathbb{Z}_2} \rangle$ the verbal ideal of the free associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra of the rank ν generated by all appropriate evaluations of elements of Γ_1 .

- Remark 1**
1. $f(x_1, \dots, x_n) \in \text{Id}^{\mathbb{Z}_2}(F\langle X_\nu^{\mathbb{Z}_2} \rangle / (\Gamma \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle))$ if and only if $f(h_1, \dots, h_n) \in \Gamma$ for all homogeneous polynomials of appropriate graded degrees $h_1, \dots, h_n \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$.
 2. Let A be a finitely generated associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra, $\Gamma_2 = \text{Id}^{\mathbb{Z}_2}(A)$ the ideal of its graded identities, and $\nu \geq \text{grk}(A)$. Then $\Gamma_1(F\langle X_\nu^{\mathbb{Z}_2} \rangle) \subseteq \Gamma_2$ implies that $\Gamma_1 \subseteq \Gamma_2$.

It's well known due to the linearization process and possibility to identify variables in case of zero characteristic that any system of identities (ordinary or $\mathbb{Z}/2\mathbb{Z}$ -graded) is equivalent to a system of multilinear identities. Thus in case of zero characteristic it is enough to consider only multilinear identities.

2 Graded algebras.

Let \tilde{F} be an algebraically closed field. Consider a finite dimensional \tilde{F} -superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$. Then we have an analog of the Wedderburn-Maltsev decomposition for superalgebras.

Lemma 1 *Let \tilde{F} be an algebraically closed field. Any finite dimensional \tilde{F} -superalgebra A is isomorphic as a superalgebra to an F -superalgebra of the form*

$$A' = (C_1 \times \dots \times C_p) \oplus J. \quad (1)$$

Where the Jacobson radical $J = J(A')$ of A' is a graded ideal, $B' = C_1 \times \dots \times C_p$ is a maximal graded semisimple subalgebra of A' , $p \in \mathbb{N} \cup \{0\}$. A $\mathbb{Z}/2\mathbb{Z}$ -graded simple component C_l is one of the superalgebras $M_{k_l, m_l}(\tilde{F})$ (a matrix algebra with an elementary $\mathbb{Z}/2\mathbb{Z}$ -grading) or $M_{k_l}(\tilde{F}[c])$ (a matrix algebra over the group algebra of $\tilde{F}[c | c^2 = 1]$ with the grading induced by the natural grading of the group algebra).

Moreover $A' = \text{Span}_{\tilde{F}}\{D, U\}$, where

$$\begin{aligned}
D &= \cup_{1 \leq l \leq p} D_l, \\
D_l &= \{E_{l i_l j_l} = \varepsilon_l E_{l i_l j_l} \varepsilon_l \mid 1 \leq i_l, j_l \leq s_l\}, \\
s_l &= k_l + m_l \quad \text{if } C_l = M_{k_l, m_l}(\tilde{F}); \\
D_l &= \{E_{l i_l j_l} = \varepsilon_l E_{l i_l j_l} \varepsilon_l, E_{l i_l j_l} c = \varepsilon_l E_{l i_l j_l} c \varepsilon_l \mid 1 \leq i_l, j_l \leq s_l\}, \quad (2) \\
s_l &= k_l, c^2 = 1 \quad \text{if } C_l = M_{k_l}(\tilde{F}[c]); \\
U &= \{\varepsilon_{l'} u_\theta \varepsilon_{l''} \mid l', l'' = 1, \dots, p+1; u_\theta \in J_\theta = J \cap A'_\theta, \theta \in \mathbb{Z}/2\mathbb{Z}\} \quad (3)
\end{aligned}$$

are the sets of homogeneous in the grading elements. $E_{l i_l j_l}$ is the matrix unit. Elements of D are homogeneous with the grading defined by the next equalities $\deg_{\mathbb{Z}_2} E_{l i_l j_l} = \bar{0}$ if $1 \leq i_l, j_l \leq k_l$, or $k_l + 1 \leq i_l, j_l \leq k_l + m_l$, and $\deg_{\mathbb{Z}_2} E_{l i_l j_l} = \bar{1}$ otherwise in $C_l = M_{k_l, m_l}(\tilde{F})$; $\deg_{\mathbb{Z}_2} E_{l i_l j_l} = \bar{0}$, $\deg_{\mathbb{Z}_2} E_{l i_l j_l} \cdot c = \bar{1}$ for all $1 \leq i_l, j_l \leq k_l$ in $C_l = M_{k_l}(\tilde{F}[c])$. The element ε_l is a minimal orthogonal central idempotent of B' , homogeneous in the grading, that corresponds to the unit element of C_l ($l = 1, \dots, p$). Elements u_θ runs on the set of homogeneous basic elements of the Jacobson radical $J = J(A) = \oplus_{l', l''=1}^{p+1} \varepsilon_{l'} J \varepsilon_{l''}$, where ε_{p+1} is the adjoint idempotent.

Proof. It is a classical result (see, e.g., [9] or [2], [3] for a more general case) that a finite dimensional superalgebra A can be represented as $B \oplus J$, where J is the Jacobson radical of A , that is a graded ideal, and B is a maximal $\mathbb{Z}/2\mathbb{Z}$ -graded semisimple subalgebra of A . It is also well known that $B \cong \times_{l=1}^p C_l$, where C_l are simple superalgebras. C_l is isomorphic either to a matrix superalgebra $M_{k_l, m_l}(F) = M_{k_l+m_l}(F)$ with an elementary grading

$$(C_l)_{\bar{0}} = \left\{ \begin{array}{cc} k_l & m_l \\ * & 0 \\ 0 & * \end{array} \right\}, \quad (C_l)_{\bar{1}} = \left\{ \begin{array}{cc} k_l & m_l \\ 0 & * \\ * & 0 \end{array} \right\},$$

or to a matrix algebra over the group algebra $M_{k_l}(\tilde{F}[\mathbb{Z}/2\mathbb{Z}]) \simeq M_{k_l}(\tilde{F}[c|c^2=1])$ with the grading induced by the natural $\mathbb{Z}/2\mathbb{Z}$ -grading of the group algebra $(C_l)_{\bar{0}} = M_{k_l}(\tilde{F})$, $(C_l)_{\bar{1}} = M_{k_l}(\tilde{F}) \cdot c$.

Thus $A' = B' \oplus J$ can be assumed our superalgebra. Any $\mathbb{Z}/2\mathbb{Z}$ -simple component C_l of the semisimple graded subalgebra B' has the unit element of the even degree. It gives the minimal orthogonal central idempotent ε_l of the algebra B' . D is a homogeneous basis of B' . It is clear that any element $r \in J$ can be uniquely represented as a sum of elements of graded subspaces $\varepsilon_{l'} J \varepsilon_{l''}$, $l', l'' = 1, \dots, p+1$. Moreover $\varepsilon_l a = 0$ (and $a \varepsilon_l = 0$) for any $a \in \varepsilon_{l'} J \varepsilon_{l''}$ with $l \neq l'$ ($l \neq l''$), $l = 1, \dots, p$. \square

The next construction is useful to extend the previous result for any field of zero characteristic in some sense.

Take a superalgebra B (not necessarily without unit). We denote by $B^\# = B \oplus F \cdot 1_F$ the superalgebra with the adjoint unit 1_F .

Let us take a finite dimensional superalgebra $A = B \oplus J(A)$ with a maximal $\mathbb{Z}/2\mathbb{Z}$ -graded semisimple subalgebra B and the Jacobson radical $J(A)$. Consider a $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebra $\tilde{B} \subseteq B$, and a positive integer number q . Consider the free product $\tilde{B}^\# *_F F\langle X_q^{\mathbb{Z}_2} \rangle^\#$, and define on it the $\mathbb{Z}/2\mathbb{Z}$ -grading by the equalities $\deg_{\mathbb{Z}_2}(u_1 \cdots u_s) = (\deg_{\mathbb{Z}_2} u_1) + \cdots + (\deg_{\mathbb{Z}_2} u_s)$, where $u_i \in \tilde{B}^\# \cup F\langle X_q^{\mathbb{Z}_2} \rangle^\#$ are homogeneous elements. Let $\tilde{B}(X_q^{\mathbb{Z}_2})$ be the graded subalgebra of $\tilde{B}^\# *_F F\langle X_q^{\mathbb{Z}_2} \rangle^\#$ generated by the set $\tilde{B} \cup F\langle X_q^{\mathbb{Z}_2} \rangle$. Denote by $(X_q^{\mathbb{Z}_2})$ the two-sided graded ideal of $\tilde{B}(X_q^{\mathbb{Z}_2})$ generated by the set of variables $X_q^{\mathbb{Z}_2}$. Particularly, it is clear that $\tilde{B}(X_q^{\mathbb{Z}_2}) = \tilde{B} \oplus (X_q^{\mathbb{Z}_2})$.

Given a \mathbb{Z}_2 -ideal Γ denote by $\Gamma(\tilde{B}(X_q^{\mathbb{Z}_2}))$ the two-sided graded verbal ideal of $\tilde{B}(X_q^{\mathbb{Z}_2})$ generated by results of all appropriate evaluations of polynomials from Γ . Take any $s \in \mathbb{N}$, and consider the quotient algebra

$$\mathcal{R}_{q,s}(\tilde{B}, \Gamma) = \tilde{B}(X_q^{\mathbb{Z}_2}) / (\Gamma(\tilde{B}(X_q^{\mathbb{Z}_2})) + (X_q^{\mathbb{Z}_2})^s). \quad (4)$$

Denote also $\mathcal{R}_{q,s}(A) = \mathcal{R}_{q,s}(B, \text{Id}^{\mathbb{Z}_2}(A))$ for $\Gamma = \text{Id}^{\mathbb{Z}_2}(A)$, $\tilde{B} = B$.

Lemma 2 *Take any natural numbers q, s and a \mathbb{Z}_2 -ideal Γ such that $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(A)$. The algebra $\mathcal{R}_{q,s}(\tilde{B}, \Gamma)$ is a finite dimensional superalgebra with the ideal of graded identities $\text{Id}^{\mathbb{Z}_2}(\mathcal{R}_{q,s}(\tilde{B}, \Gamma)) \supseteq \Gamma$. Moreover $\mathcal{R}_{q,s}(\tilde{B}, \Gamma) = \overline{B} \oplus J(\mathcal{R}_{q,s}(\tilde{B}, \Gamma))$. Here \overline{B} is a maximal semisimple $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebra of $\mathcal{R}_{q,s}(\tilde{B}, \Gamma)$, and $\overline{B} \cong \tilde{B}$. The Jacobson radical of $\mathcal{R}_{q,s}(\tilde{B}, \Gamma)$ is equal to $(X_q^{\mathbb{Z}_2}) / (\Gamma(\tilde{B}(X_q^{\mathbb{Z}_2})) + (X_q^{\mathbb{Z}_2})^s)$, and is nilpotent of degree less or equal to s .*

If $q \geq \text{grk}(J(A))$, and $s \geq \text{nd}(A)$ then $\text{Id}^{\mathbb{Z}_2}(\mathcal{R}_{q,s}(A)) = \text{Id}^{\mathbb{Z}_2}(A)$.

Proof. It is clear that $I = \Gamma(\tilde{B}(X_q^{\mathbb{Z}_2})) + (X_q^{\mathbb{Z}_2})^s \subseteq (X_q^{\mathbb{Z}_2})$, and $\tilde{B} \cap I = (0)$. I is a graded ideal of $\tilde{B}(X_q^{\mathbb{Z}_2})$. Hence $\mathcal{R}_{q,s}(\tilde{B}, \Gamma) = \tilde{B}(X_q^{\mathbb{Z}_2}) / I$ is a superalgebra. Then for the canonical homomorphism $\psi : \tilde{B}(X_q^{\mathbb{Z}_2}) \rightarrow \mathcal{R}_{q,s}(\tilde{B}, \Gamma)$ we obtain $\overline{B} = \psi(\tilde{B}) \cong \tilde{B}$. Hence $\mathcal{R}_{q,s}(\tilde{B}, \Gamma) = \overline{B} \oplus \psi((X_q^{\mathbb{Z}_2}))$, where $\psi((X_q^{\mathbb{Z}_2})) = (X_q^{\mathbb{Z}_2}) / I$ is the maximal nilpotent ideal of the algebra $\mathcal{R}_{q,s}(\tilde{B}, \Gamma)$ of degree at most s . It is clear that $(X_q^{\mathbb{Z}_2}) / I$ is $\mathbb{Z}/2\mathbb{Z}$ -graded and finite dimensional. Then $\mathcal{R}_{q,s}(\tilde{B}, \Gamma)$ is also a finite dimensional algebra with the Jacobson radical $J(\mathcal{R}_{q,s}(\tilde{B}, \Gamma)) = \psi((X_q^{\mathbb{Z}_2}))$. It is clear that $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{R}_{q,s}(\tilde{B}, \Gamma))$ for any $q, s \in \mathbb{N}$.

Let us take $\Gamma = \text{Id}^{\mathbb{Z}_2}(A)$, $\tilde{B} = B$. Suppose that the Jacobson radical $J(A)$ of the algebra A is generated as an algebra by the set $\{r_1, \dots, r_\nu\}$. Then we have $r_i = \sum_{\theta \in \mathbb{Z}/2\mathbb{Z}} r_{i\theta}$, where $r_{i\theta} \in J(A) \cap A_\theta$ ($i = 1, \dots, \nu$). Consider the map $\varphi : \overline{x}_{i\theta} = x_{i\theta} + I \mapsto r_{i\theta}$ ($i = 1, \dots, \nu$). Assume that $\varphi(b + I) = b$ for any $b \in B$. If $q \geq \nu$, and $s \geq \text{nd}(A)$ then φ can be extended to a surjective graded homomorphism $\varphi : \mathcal{R}_{q,s}(A) \rightarrow A$. Therefore $\Gamma = \text{Id}^{\mathbb{Z}_2}(A) \supseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{R}_{q,s}(A))$. \square

Definition 2 *F-algebra A is called representable if A can be embedded into some algebra C that is finite dimensional over an extension $\tilde{F} \supseteq F$ of the base field F .*

Lemma 3 *If a $\mathbb{Z}/2\mathbb{Z}$ -graded F -algebra A is representable then there exists a finite dimensional over F $\mathbb{Z}/2\mathbb{Z}$ -graded F -algebra U such that $\text{Id}^{\mathbb{Z}_2}(A) = \text{Id}^{\mathbb{Z}_2}(U)$.*

Proof. Suppose that A is isomorphic to an F -subalgebra \mathcal{B} of a finite dimensional \tilde{F} -algebra $\tilde{\mathcal{B}}$. We can assume that the extension $\tilde{F} \supseteq F$ is algebraically closed, and \mathcal{B} is $\mathbb{Z}/2\mathbb{Z}$ -graded. Consider the algebra $\tilde{U} = \tilde{U}_0 \oplus \tilde{U}_1$, where $\tilde{U}_\theta = (\tilde{F}\mathcal{B}_\theta) \otimes_{\tilde{F}} \tilde{F}\theta \subseteq \tilde{\mathcal{B}} \otimes_{\tilde{F}} \tilde{F}[\mathbb{Z}/2\mathbb{Z}]$. \tilde{U} is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded \tilde{F} -algebra. And $\text{Id}^{\mathbb{Z}_2}(\tilde{U}) = \text{Id}^{\mathbb{Z}_2}(\mathcal{B}) = \text{Id}^{\mathbb{Z}_2}(A)$. Let $\tilde{C}_l = \tilde{C}_{l0} \oplus \tilde{C}_{l1}$ be a $\mathbb{Z}/2\mathbb{Z}$ -simple component in the decomposition (1) of the algebra \tilde{U} .

It is clear from the classification of simple superalgebras (see also Lemma 1) that \tilde{C}_l contains a finite dimensional over F $\mathbb{Z}/2\mathbb{Z}$ -graded simple F -subalgebra $C_l = C_{l0} \oplus C_{l1}$ satisfying $\tilde{C}_{l\theta} = \tilde{F}C_{l\theta}$ for any $\theta \in \mathbb{Z}/2\mathbb{Z}$ ($l = 1, \dots, p$). More precisely, $C_l = M_{k_l, m_l}(F)$ if $\tilde{C}_l = M_{k_l, m_l}(\tilde{F})$, and $C_l = M_{k_l}(F[c])$ if $\tilde{C}_l = M_{k_l}(\tilde{F}[c])$. Moreover the algebras \tilde{C}_l and C_l have the same canonic homogeneous base of type (2) over \tilde{F} and F respectively.

Let us take $B = C_1 \times \dots \times C_p$, $\Gamma = \text{Id}^{\mathbb{Z}_2}(A)$, $q = \dim_{\tilde{F}} J(\tilde{U})$, $s = \text{nd}(\tilde{U})$. Then the F -algebra $U = \mathcal{R}_{q,s}(B, \Gamma)$ defined by (4) is $\mathbb{Z}/2\mathbb{Z}$ -graded finite dimensional over F . And $\text{Id}^{\mathbb{Z}_2}(U) = \Gamma = \text{Id}^{\mathbb{Z}_2}(\tilde{U})$ (Lemma 2). \square

Definition 3 *We say that an F -finite dimensional superalgebra A' has an elementary decomposition if it satisfies the assertion of Lemma 1.*

It is clear that the direct product of superalgebras with elementary decomposition is the superalgebra with elementary decomposition. Also if F is algebraically closed then any finite dimensional F -superalgebra has an elementary decomposition.

Corollary 4 *Let F be a field of characteristic zero. Any finite dimensional F -superalgebra is \mathbb{Z}_2 PI-equivalent to a finite dimensional F -superalgebra with elementary decomposition.*

Proof. A finite dimensional F -superalgebra A can be naturally embedded to the superalgebra $\tilde{A} = A \otimes_F \tilde{F}$ preserving graded identities. We assume here that $\deg_{\mathbb{Z}_2} a \otimes \alpha = \deg_G a$, for all $a \in A$, $\alpha \in \tilde{F}$. The superalgebra \tilde{A} is finite dimensional over \tilde{F} . By Lemma 3 there exists a finite dimensional F -superalgebra A' with elementary decomposition such that $\text{Id}^{\mathbb{Z}_2}(A') = \text{Id}^{\mathbb{Z}_2}(\tilde{A}) = \text{Id}^{\mathbb{Z}_2}(A)$. Where all identities are considered over the field F . \square

Therefore considering graded identities of a finite dimensional superalgebra A we always can assume that A has a form (1), and a basis of A can be chosen in the set $D \cup U$. Particularly, for multilinear graded polynomials it is enough to consider only evaluations by elements of the set $D \cup U$. Such evaluation of a multilinear graded polynomial is called *elementary*. Elements of the set D are called semisimple, and elements of the set U are radical.

Definition 4 *Let $A = B \oplus J$ be a finite dimensional superalgebra, $B = \bigoplus_{\theta \in \mathbb{Z}/2\mathbb{Z}} B_\theta$ a maximal semisimple $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebra of A , and $J(A) = J$ the Jacobson*

radical of A . We denote by $\dim_{\mathbb{Z}_2} A = (\dim B_{\bar{0}}, \dim B_{\bar{1}})$, and by $\text{nd}(A)$ the nilpotency degree of the radical J . Consider as principal the next parameter of A $\text{par}_{\mathbb{Z}_2}(A) = (\dim_{\mathbb{Z}_2} A; \text{nd}(A))$.

The 4-tuple $\text{cpar}_{\mathbb{Z}_2}(A) = (\text{par}_{\mathbb{Z}_2}(A); \dim J(A))$ is called complex parameter of A .

A finite dimensional superalgebra A is nilpotent if and only if $\dim_{\mathbb{Z}_2} A = (0, 0)$. Recall that n -tuples of numbers are ordered lexicographically. Then for any nonzero graded two-side ideal $I \trianglelefteq A$ we have that $\text{cpar}_{\mathbb{Z}_2}(A/I) < \text{cpar}_{\mathbb{Z}_2}(A)$.

3 Kemer index.

Let $f = f(y_1, \dots, y_k, x_1, \dots, x_n) \in F\langle X^{\mathbb{Z}_2} \rangle$ be a polynomial linear in all variables of the set $Y = \{y_1, \dots, y_k\}$. The polynomial f is alternating in Y , if

$$f(y_{\sigma(1)}, \dots, y_{\sigma(k)}, x_1, \dots, x_n) = (-1)^\sigma f(y_1, \dots, y_k, x_1, \dots, x_n)$$

holds for any permutation $\sigma \in \text{Sym}_k$.

For any polynomial $g(y_1, \dots, y_k, x_1, \dots, x_n)$ that is linear in $Y = \{y_1, \dots, y_k\}$, it is possible to construct a polynomial alternating in Y by setting

$$f(y_1, \dots, y_k, x_1, \dots, x_n) = \mathcal{A}_Y(g) = \sum_{\sigma \in \text{Sym}_k} (-1)^\sigma g(y_{\sigma(1)}, \dots, y_{\sigma(k)}, x_1, \dots, x_n).$$

The corresponding mapping \mathcal{A}_Y is a linear transformation, we called it the alternator. Any polynomial f alternating in Y can be decomposed as $f = \sum_{i=1}^s \alpha_i \mathcal{A}_Y(u_i)$, where u_i s are monomials of f , $\alpha_i \in F$. The properties of graded alternating polynomials are similar to that of non-graded polynomials. We consider only polynomials alternating in homogeneous sets of variables.

Given a pair $\bar{t} = (t_{\bar{0}}, t_{\bar{1}}) \in \mathbb{N}_0^2$ we say that a graded polynomial $f \in F\langle X^{\mathbb{Z}_2} \rangle$ has a collection of \bar{t} -alternating graded variables (f is \bar{t} -alternating) if $f(Y_{\bar{0}}, Y_{\bar{1}}, X)$ is linear in $Y = Y_{\bar{0}} \cup Y_{\bar{1}}$, and f is alternating in each set $Y_\theta = \{y_{1\theta}, \dots, y_{t_\theta\theta}\} \subseteq X_\theta$, $|Y_\theta| = t_\theta$, $\theta \in \mathbb{Z}/2\mathbb{Z}$.

Definition 5 Fix any $\bar{t} = (t_{\bar{0}}, t_{\bar{1}}) \in \mathbb{N}_0^2$. Suppose that $\tau_1, \dots, \tau_s \in \mathbb{N}_0^2$ are some (possibly different) pairs satisfying the conditions $\tau_j = (\tau_{j\bar{0}}, \tau_{j\bar{1}}) > \bar{t} = (t_{\bar{0}}, t_{\bar{1}})$ for all $j = 1, \dots, s$. Let $f \in F\langle X^{\mathbb{Z}_2} \rangle$ be a multihomogeneous graded polynomial. Suppose that $f = f(Y_1, \dots, Y_{s+\mu}; X)$ has s collections of τ_j -alternating variables $Y_j = Y_{j\bar{0}} \cup Y_{j\bar{1}}$ ($j = 1, \dots, s$), and μ collections of \bar{t} -alternating variables $Y_j = Y_{j\bar{0}} \cup Y_{j\bar{1}}$ ($j = s+1, \dots, s+\mu$). We assume that all these sets are disjoint. Then we say that f is of the type $(\bar{t}; s; \mu) = (t_1, t_2; s; \mu)$. Here $Y_{j\theta} \subseteq X_\theta$ with $|Y_{j\theta}| = \tau_{j\theta}$ for any $j = 1, \dots, s$, and $|Y_{j\theta}| = t_\theta$ for any $j = s+1, \dots, s+\mu$.

Observe that a multihomogeneous polynomial f of a type $(t; s; \mu)$ is also of the type $(t; s'; \mu')$ for all $s' \leq s$, and $\mu' \leq \mu$. Particularly, any nontrivial graded multilinear polynomial of degree s has the type $(0, 0; s; \mu)$ for any $\mu \in \mathbb{N}_0$.

Definition 6 Given a $\mathbb{Z}_2 T$ -ideal $\Gamma \trianglelefteq F\langle X^{\mathbb{Z}_2} \rangle$ the parameter $\beta(\Gamma) = (t_{\bar{0}}, t_{\bar{1}})$ is the greatest lexicographic pair $\bar{t} = (t_{\bar{0}}, t_{\bar{1}}) \in \mathbb{N}_0^2$ such that for any $s \in \mathbb{N}$ there exists a graded polynomial $f \notin \Gamma$ of the type $(\bar{t}; 0; s)$.

The parameter $\beta(\Gamma)$ is well defined for any proper $\mathbb{Z}_2 T$ -ideal $\Gamma \trianglelefteq F\langle X^{\mathbb{Z}_2} \rangle$ of a finitely generated $\mathbb{Z}/2\mathbb{Z}$ -graded PI-algebra. In this case Γ contains the ordinary Capelli polynomial of some order d ([11]). Hence any graded polynomial f of the type $(t_{\bar{0}}, t_{\bar{1}}; 0; s)$ belongs to Γ if $t_{\bar{0}} \geq d$ or $t_{\bar{1}} \geq d$. The next parameter is also well defined.

Definition 7 Given a nonnegative integer μ let $\gamma(\Gamma; \mu) = s \in \mathbb{N}$ be the smallest integer $s > 0$ such that any graded polynomial of the type $(\beta(\Gamma); s; \mu)$ belongs to Γ .

$\gamma(\Gamma; \mu)$ is a positive non-increasing function of μ . Let us denote the limit of this function by $\gamma(\Gamma) = \lim_{\mu \rightarrow \infty} \gamma(\Gamma; \mu) \in \mathbb{N}$. Then $\omega(\Gamma)$ is the smallest number $\hat{\mu}$ such that $\gamma(\Gamma; \mu) = \gamma(\Gamma)$ for any $\mu \geq \hat{\mu}$.

Definition 8 We call by the Kemer index of a $\mathbb{Z}_2 T$ -ideal Γ the lexicographically ordered collection $\text{ind}_{\mathbb{Z}_2}(\Gamma) = (\beta(\Gamma); \gamma(\Gamma))$.

Notice that $\text{ind}_{\mathbb{Z}_2}(\Gamma)$ is greater than $(0, 0; 1)$ for any proper $\mathbb{Z}_2 T$ -ideal Γ . We assume also that $\text{ind}_{\mathbb{Z}_2}(F\langle X^{\mathbb{Z}_2} \rangle) = (0, 0; 1)$.

Let us denote $\omega(A) = \omega(\text{Id}^{\mathbb{Z}_2}(A))$, $\gamma(A; \mu) = \gamma(\text{Id}^{\mathbb{Z}_2}(A); \mu)$, $\text{ind}_{\mathbb{Z}_2}(A) = \text{ind}_{\mathbb{Z}_2}(\text{Id}^{\mathbb{Z}_2}(A))$ for a finitely generated PI-superalgebra A .

It is clear that A is a nilpotent superalgebra of class s if and only if $\text{ind}_{\mathbb{Z}_2}(A) = (0, 0; s)$. Particularly $\text{ind}_{\mathbb{Z}_2}(A) = \text{par}_{\mathbb{Z}_2}(A)$ for a nilpotent algebra A . In general case we have

Lemma 5 $\text{ind}_{\mathbb{Z}_2}(A) \leq \text{par}_{\mathbb{Z}_2}(A)$ for any finite dimensional superalgebra A .

Proof. Let us denote $\text{dims}_{\mathbb{Z}_2} A = (t_{\bar{0}}, t_{\bar{1}})$. Suppose that $\beta_{\tilde{\theta}} > t_{\tilde{\theta}}$ for some $\tilde{\theta} = \bar{0}, \bar{1}$, and a multilinear polynomial f has the type $(\beta_{\bar{0}}, \beta_{\bar{1}}; 0; \text{nd}(A))$. Then for any $j = 1, \dots, \text{nd}(A)$ all $\tilde{\theta}$ -variables of the alternating set $Y_{j\tilde{\theta}}$ of f can not be evaluated only by semisimple elements with nonzero result. Therefore $f \in \text{Id}^{\mathbb{Z}_2}(A)$, and $\text{ind}_{\mathbb{Z}_2}(A) \leq \text{par}_{\mathbb{Z}_2}(A)$. \square

Definition 9 Given a $\mathbb{Z}_2 T$ -ideal Γ , and any $\mu \in \mathbb{N}_0$ a multihomogeneous polynomial $f \in F\langle X^{\mathbb{Z}_2} \rangle$ is called μ -boundary for Γ if $f \notin \Gamma$, and f has the type $(\beta(\Gamma); \gamma(\Gamma) - 1; \mu)$.

Let us denote by $S_{\mu}(\Gamma)$ the set of all μ -boundary polynomials for Γ . Denote also $S_{\mu}(A) = S_{\mu}(\text{Id}^{\mathbb{Z}_2}(A))$, $K_{\mu}(\Gamma) = \mathbb{Z}_2 T[S_{\mu}(\Gamma)]$, $K_{\mu, A} = K_{\mu}(\text{Id}^{\mathbb{Z}_2}(A)) = \mathbb{Z}_2 T[S_{\mu}(A)]$.

Observe that if the Kemer index is well defined for a $\mathbb{Z}_2 T$ -ideal Γ then Γ has multilinear boundary polynomials for all $\mu \in \mathbb{N}_0$. Moreover a polynomial f belongs to $S_{\mu}(\Gamma)$ if and only if its full multilinearization \tilde{f} belongs to $S_{\mu}(\Gamma)$. Definitions 6, 7 immediately imply the next basic properties of the Kemer index and boundary polynomials.

Lemma 6 Given \mathbb{Z}_2 -ideals Γ_1, Γ_2 admitting the Kemer index if $\Gamma_1 \subseteq \Gamma_2$ then $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) \geq \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$.

Lemma 7 Consider \mathbb{Z}_2 -ideals $\Gamma, \Gamma_1, \dots, \Gamma_\rho$ admitting the Kemer index. Assume that $\text{ind}_{\mathbb{Z}_2}(\Gamma_i) < \text{ind}_{\mathbb{Z}_2}(\Gamma)$ for all $i = 1, \dots, \rho$. Then there exists $\hat{\mu} \in \mathbb{N}_0$ such that $S_\mu(\Gamma) \subseteq \bigcap_{i=1}^{\rho} \Gamma_i$ for any $\mu \geq \hat{\mu}$.

Proof. Let us denote $\text{ind}_{\mathbb{Z}_2}(\Gamma_i) = (\beta_i, \gamma_i)$, $1 \leq i \leq \rho$, $\text{ind}_{\mathbb{Z}_2}(\Gamma) = (\beta, \gamma)$. Assume that $\beta_i < \beta$ for $i = 1, \dots, \rho'$, and $\beta_i = \beta$, $\gamma_i < \gamma$ for $i = \rho' + 1, \dots, \rho$ ($0 \leq \rho' \leq \rho$).

If $\beta > \beta_i$ ($1 \leq i \leq \rho'$) then there exists μ_i such that any polynomial of the type $(\beta; 0; \mu_i)$ belongs to Γ_i . If $\beta_i = \beta$ and $\gamma_i < \gamma$ ($i = \rho' + 1, \dots, \rho$) then any polynomial of the type $(\beta; \gamma_i; \mu_i)$ belongs to Γ_i for all $\mu_i \geq \omega(\Gamma_i)$. Thus $S_\mu(\Gamma) \subseteq \bigcap_{i=1}^{\rho} \Gamma_i$ for any $\mu \geq \max\{\mu_1, \dots, \mu_{\rho'}, \omega(\Gamma_{\rho'+1}), \dots, \omega(\Gamma_\rho)\}$. \square

Lemmas 6, 7 jointly give the next properties.

Lemma 8 Given \mathbb{Z}_2 -ideals Γ_1, Γ_2 admitting the Kemer index $\text{ind}_{\mathbb{Z}_2}(\Gamma_1 \cap \Gamma_2) = \max_{i=1,2} \text{ind}_{\mathbb{Z}_2}(\Gamma_i)$.

Lemma 9 For all finitely generated PI-superalgebras A_i $\text{ind}_{\mathbb{Z}_2}(A_1 \times \dots \times A_\rho) = \max_{1 \leq i \leq \rho} \text{ind}_{\mathbb{Z}_2}(A_i)$.

Lemma 10 Given \mathbb{Z}_2 -ideals Γ_1, Γ_2 admitting the Kemer index, and satisfying $\Gamma_1 \subseteq \Gamma_2$ one of the following alternatives takes place:

1. $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) = \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$, and $S_\mu(\Gamma_1) \supseteq S_\mu(\Gamma_2)$, $K_\mu(\Gamma_1) \supseteq K_\mu(\Gamma_2) \quad \forall \mu \in \mathbb{N}_0$;
2. $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) > \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$, and $S_{\hat{\mu}}(\Gamma_1) \subseteq \Gamma_2$ for some $\hat{\mu} \in \mathbb{N}_0$.

Moreover in the case $\Gamma_1 \subseteq \Gamma_2$ the conditions $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) > \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$ and $S_\mu(\Gamma_1) \subseteq \Gamma_2$ are equivalent for some $\mu \in \mathbb{N}_0$.

Proof. By Lemma 6 we have that $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) \geq \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$. The conditions $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) = \text{ind}_{\mathbb{Z}_2}(\Gamma_2) = (\beta, \gamma)$, $\Gamma_1 \subseteq \Gamma_2$ imply that $S_\mu(\Gamma_1) \supseteq S_\mu(\Gamma_2)$, $K_\mu(\Gamma_1) \supseteq K_\mu(\Gamma_2) \quad \forall \mu \in \mathbb{N}_0$.

If $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) > \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$ then $S_{\hat{\mu}}(\Gamma_1) \subseteq \Gamma_2$ holds for some $\hat{\mu} \in \mathbb{N}_0$ by Lemma 7. And visa versa if $\Gamma_1 \subseteq \Gamma_2$, and $S_{\hat{\mu}}(\Gamma_1) \subseteq \Gamma_2$ for some $\hat{\mu} \in \mathbb{N}_0$ then $S_{\hat{\mu}}(\Gamma_2) \subseteq S_{\hat{\mu}}(\Gamma_1) \subseteq \Gamma_2$ gives a contradiction. Therefore in this case we obtain that $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) > \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$. \square

Similarly the conditions $\text{ind}_{\mathbb{Z}_2}(\Gamma_1) = \text{ind}_{\mathbb{Z}_2}(\Gamma_2)$ and $S_\mu(\Gamma_1) \supseteq S_\mu(\Gamma_2)$ are also equivalent in the case $\Gamma_1 \subseteq \Gamma_2$.

The last lemma has the following corollary.

Lemma 11 *Given an integer $\mu \in \mathbb{N}_0$ and finitely generated PI-superalgebras A_1, \dots, A_ρ with the same Kemer index $\text{ind}_{\mathbb{Z}_2}(A_i) = \kappa$ for all $i = 1, \dots, \rho$ it holds*

$$S_\mu(A_1 \times \dots \times A_\rho) = S_\mu\left(\bigcap_{i=1}^{\rho} \text{Id}^{\mathbb{Z}_2}(A_i)\right) = \bigcup_{i=1}^{\rho} S_\mu(A_i),$$

$$K_{\mu, A_1 \times \dots \times A_\rho} = K_\mu\left(\bigcap_{i=1}^{\rho} \text{Id}^{\mathbb{Z}_2}(A_i)\right) = \sum_{i=1}^{\rho} K_\mu(\text{Id}^{\mathbb{Z}_2}(A_i)) = \sum_{i=1}^{\rho} K_{\mu, A_i}.$$

Lemma 12 *Given a \mathbb{Z}_2 T-ideal Γ , and a non-positive integer $\mu \in \mathbb{N}_0$ we have that $\text{ind}_{\mathbb{Z}_2}(\Gamma) > \text{ind}_{\mathbb{Z}_2}(\Gamma + K_\mu(\Gamma))$.*

Proof. Since $\Gamma \subseteq \Gamma + K_\mu(\Gamma)$ and $S_\mu(\Gamma) \subseteq K_\mu(\Gamma) \subseteq \Gamma + K_\mu(\Gamma)$ then the assertion immediately follows from Lemma 10. \square

4 \mathbb{Z}_2 PI-reduced algebras.

Definition 10 *A finite dimensional superalgebra A with elementary decomposition is \mathbb{Z}_2 PI-reduced if there do not exist finite dimensional superalgebras A_1, \dots, A_ϱ with elementary decomposition such that $\bigcap_{i=1}^{\varrho} \text{Id}^{\mathbb{Z}_2}(A_i) = \text{Id}^{\mathbb{Z}_2}(A)$, and $\text{cpar}_{\mathbb{Z}_2}(A_i) < \text{cpar}_{\mathbb{Z}_2}(A)$ for all $i = 1, \dots, \varrho$.*

It is clear that a nilpotent finite dimensional superalgebra A is \mathbb{Z}_2 PI-reduced if and only if it has the minimal dimension among all nilpotent finite dimensional superalgebras satisfying the same graded identities as A .

Lemma 13 *Any simple finite dimensional superalgebra A with elementary decomposition is \mathbb{Z}_2 PI-reduced, and $\text{ind}_{\mathbb{Z}_2}(A) = \text{par}_{\mathbb{Z}_2}(A) = (t_{\bar{0}}, t_{\bar{1}}; 1)$ (i.e. $\beta(A) = \text{dims}_{\mathbb{Z}_2} A$, and $\gamma(A) = \text{nd}(A) = 1$).*

Proof. Suppose that $\text{dims}_{\mathbb{Z}_2} A = (t_{\bar{0}}, t_{\bar{1}})$. Any finite dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded simple algebra is semisimple, $\text{nd}(A) = 1$, and $\text{cpar}_{\mathbb{Z}_2}(A) = (\text{dims}_{\mathbb{Z}_2} A; 1; 0)$. It follows from Lemma 5 that $\beta(A) \leq \text{dims}_{\mathbb{Z}_2} A = (t_{\bar{0}}, t_{\bar{1}})$. Hence it is enough to construct a polynomial of the type $(t_{\bar{0}}, t_{\bar{1}}; 0; \hat{s})$ that is not a graded identity of A for any $\hat{s} \in \mathbb{N}$. Fix any natural number \hat{s} . Since A has an elementary decomposition then $A = M_{k,m}(F)$ or $A = M_k(F[c])$.

Consider the case $A = M_{k,m}(F)$ at first. In this case A has a homogeneous in the grading basis of the type $\{E_{ij} | i, j = 1, \dots, k+m\}$, where E_{ij} are the matrix units. Consider \hat{s} sets of $(k+m)^2$ distinct graded variables $Y_d = \{y_{d,(ij)} \in X^{\mathbb{Z}_2} | i, j = 1, \dots, k+m\}$. Here any graded variable $y_{d,(ij)}$ corresponds to the matrix unit E_{ij} , and $\deg_{\mathbb{Z}_2} y_{d,(ij)} = \deg_{\mathbb{Z}_2} E_{ij}$ in the superalgebra $M_{k,m}(F)$ ($d = 1, \dots, \hat{s}$).

Denote by $Y_{d,\theta} = \{y \in Y_d | \deg_{\mathbb{Z}_2} y = \theta\}$. Then for any d we have that $|Y_{d,\theta}| = t_\theta = \dim A_\theta$, $\theta \in \mathbb{Z}/2\mathbb{Z}$. Let us take also the set of graded variables $Z = \{z_{(ij)} \in X^{\mathbb{Z}_2} | \deg_{\mathbb{Z}_2} z_{(ij)} = \deg_{\mathbb{Z}_2} E_{ij}, i, j = 1, \dots, k+m\}$. We assume that Z is disjoint with

$Y = \bigcup_{d=1}^{\hat{s}} Y_d$. We say that the variable $z_{(j_1 i_2)}$ connects the variables $y_{d, (i_1 j_1)}$ and $y_{d, (i_2 j_2)}$. Let us consider for any fixed d the graded monomial w_d that is the product of all variables $y_{d, (i_s j_s)}$ connected by variables $z_{(ij)}$

$$w_d = y_{d, (11)} z_{(11)} y_{d, (12)} z_{(21)} y_{d, (13)} \cdots y_{d, (i_1 j_1)} z_{(j_1 i_2)} y_{d, (i_2 j_2)} z_{(j_2 i_3)} y_{d, (i_3 j_3)} \cdots y_{d, (k+m-1, k+m)} z_{(k+m-1, k+m)} y_{d, (k+m, k+m)}, \quad (d = 1, \dots, \hat{s});$$

$$\text{and the monomial } W(Y, Z) = z_{(11)} \cdot \left(\prod_{d=1}^{\hat{s}} (w_d z_{(k+m1)}) \right). \quad (5)$$

Then the polynomial $f(Y, Z) = \left(\prod_{d=1}^{\hat{s}} (\mathcal{A}_{Y_{d, \bar{0}}} \mathcal{A}_{Y_{d, \bar{1}}}) \right) W(Y, Z)$ is $(t_{\bar{0}}, t_{\bar{1}})$ -alternating in any set $Y_d = Y_{d, \bar{0}} \cup Y_{d, \bar{1}}$ ($d = 1, \dots, \hat{s}$).

Notice that for any d and θ the set $Y_{d, \theta}$ contains at most 1 variable $y_{d, (ij)}$ for the same pair (i, j) . Consider the evaluation $y_{d, (ij)} = E_{ij}$, $z_{(ij)} = E_{ij} (*)$ of the polynomial f . Since the variables $z_{(ij)}$ fix the positions for indices of the variables y then $(*)$ gives a nonzero result

$$f|_{(*)} = W|_{(*)} = E_{11} \neq 0. \quad (6)$$

Similarly, we construct the polynomial $f(Y, Z)$, and the corresponding nonzero evaluation for the case $A = M_k(F[c])$. In this case A has a homogeneous in the grading basis of the type $\{E_{ij}, E_{ij}c | i, j = 1, \dots, k\}$, where E_{ij} are the matrix units, and c is the central element of A satisfying $c^2 = 1$. Here $\deg_{\mathbb{Z}_2} E_{ij} = \bar{0}$, and $\deg_{\mathbb{Z}_2} E_{ij}c = \bar{1}$ for all $i, j = 1, \dots, k$. The set $Y_d = \{y_{d, \bar{0}, (ij)}, y_{d, \bar{1}, (ij)} \in X^{\mathbb{Z}_2} | i, j = 1, \dots, k\}$ contains $2k^2$ graded variables. A graded variable $y_{d, \bar{0}, (ij)}$ corresponds to the basic element E_{ij} , and $y_{d, \bar{1}, (ij)}$ corresponds to $E_{ij} \cdot c$, $\deg_{\mathbb{Z}_2} y_{d, \theta, (ij)} = \theta$ ($d = 1, \dots, \hat{s}$). Then $Y_{d, \theta} = \{y_{d, \theta, (ij)} | i, j = 1, \dots, k\}$, and $|Y_{d, \theta}| = t_{\theta} = \dim A_{\theta} = k^2$, $\theta \in \mathbb{Z}/2\mathbb{Z}$. All connecting variables $z_{(ij)}$ have even degree. In this case the variable $z_{(j_1 i_2)}$ connects variables $y_{d, \theta, (i_1 j_1)}$ and $y_{d, \xi, (i_2 j_2)}$ for any $\theta, \xi \in \mathbb{Z}/2\mathbb{Z}$.

The graded monomial $w_{d, \theta}$ is the product of all variables $y_{d, \theta, (i_s j_s)}$ connected by variables $z_{(ij)}$ for any fixed $\theta \in \mathbb{Z}/2\mathbb{Z}$ and d

$$w_{d, \theta} = y_{d, \theta, (11)} z_{(11)} y_{d, \theta, (12)} z_{(21)} y_{d, \theta, (13)} \cdots y_{d, \theta, (i_1 j_1)} z_{(j_1 i_2)} y_{d, \theta, (i_2 j_2)} z_{(j_2 i_3)} y_{d, \theta, (i_3 j_3)} \cdots y_{d, \theta, (k-1, k)} z_{(k-1, k)} y_{d, \theta, (k, k)}, \quad (d = 1, \dots, \hat{s}, \quad \theta = \bar{0}, \bar{1}).$$

$$\text{Then the monomial } W(Y, Z) = z_{(11)} \cdot \left(\prod_{d=1}^{\hat{s}} (w_{d, \bar{0}} z_{(k1)} w_{d, \bar{1}} z_{(k1)}) \right). \quad (7)$$

Take the polynomial $f(Y, Z) = \left(\prod_{d=1}^{\hat{s}} \mathcal{A}_{Y_{d, \bar{0}}} \cdot \mathcal{A}_{Y_{d, \bar{1}}} \right) W(Y, Z)$. $f(Y, Z)$ is $(t_{\bar{0}}, t_{\bar{1}})$ -alternating in any set $Y_d = \bigcup_{\theta \in \mathbb{Z}/2\mathbb{Z}} Y_{d, \theta}$ ($d = 1, \dots, \hat{s}$).

For any d and θ the set $Y_{d, \theta}$ contains at most 1 variable $y_{d, \theta, (ij)}$ for the same pair (i, j) then the evaluation $y_{d, \bar{0}, (ij)} = E_{ij}$, $y_{d, \bar{1}, (ij)} = E_{ij}c$, $z_{(ij)} = E_{ij} (*)$ of f gives a nonzero result

$$f|_{(*)} = W|_{(*)} = E_{11} \cdot c^{k^2} \neq 0. \quad (8)$$

Notice that $c^{k^2} = c$ if k is odd, and $c^{k^2} = 1$ otherwise.

In both of the cases $f \notin \text{Id}^{\mathbb{Z}_2}(A)$, and this is the desired polynomial of the type $(t_0, t_1; 0; \hat{s})$. Therefore $\beta(A) = \dim_{\mathbb{Z}_2} A$. By Lemma 5 the conditions $\dim J(A) = 0$, and $\text{ind}_{\mathbb{Z}_2}(A) = \text{par}_{\mathbb{Z}_2}(A)$ imply that A is \mathbb{Z}_2 PI-reduced. \square

\mathbb{Z}_2 PI-reduced algebras exist and possess the next properties.

Lemma 14 *Given a \mathbb{Z}_2 PI-reduced algebra with the Wedderburn-Malcev decomposition (1) $A = (C_1 \times \cdots \times C_p) \oplus J$ we have $C_{\sigma(1)}JC_{\sigma(2)}J \cdots JC_{\sigma(p)} \neq 0$ for some $\sigma \in \text{Sym}_p$.*

Proof. Suppose that $C_{\sigma(1)}JC_{\sigma(2)}J \cdots JC_{\sigma(p)} = 0$ for any $\sigma \in \text{Sym}_p$. Consider the $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebras with elementary decomposition $A_i = (\prod_{\substack{1 \leq j \leq p \\ j \neq i}} C_j) \oplus J(A)$ of

the superalgebra A . Then we have $\text{Id}^{\mathbb{Z}_2}(A) = \bigcap_{i=1}^p \text{Id}^{\mathbb{Z}_2}(A_i)$, and $\dim_{\mathbb{Z}_2} A_i < \dim_{\mathbb{Z}_2} A$ for any $i = 1, \dots, p$. This contradicts to the definition of \mathbb{Z}_2 PI-reducible algebra. \square

Particularly we have $\text{nd}(A) \geq p$ for a \mathbb{Z}_2 PI-reduced algebra A .

Lemma 15 *Any finite dimensional superalgebra is \mathbb{Z}_2 PI-equivalent to a finite direct product of \mathbb{Z}_2 PI-reduced algebras.*

Proof. By Lemma 4 any finite dimensional superalgebra is \mathbb{Z}_2 PI-equivalent to a finite dimensional superalgebra A with elementary decomposition. If A is not \mathbb{Z}_2 PI-reduced then it is \mathbb{Z}_2 PI-equivalent to a finite direct product of finite dimensional elementary decomposed superalgebras with the complex parameters less than $\text{cpar}_{\mathbb{Z}_2}(A)$. We apply this process inductively to all multipliers. The set \mathbb{N}_0^4 with the lexicographical order satisfies the descending chain condition. Therefore this process of decomposition will stop after a finite number of steps. \square

Lemma 15 along with Lemmas 10, 11 implies

Lemma 16 *Any finite dimensional superalgebra A is \mathbb{Z}_2 PI-equivalent to a direct product $\mathcal{O}(A) \times \mathcal{Y}(A)$ of finite dimensional superalgebras with elementary decomposition satisfying $\text{ind}_{\mathbb{Z}_2}(A) = \text{ind}_{\mathbb{Z}_2}(\mathcal{O}(A)) > \text{ind}_{\mathbb{Z}_2}(\mathcal{Y}(A))$. Moreover $\mathcal{O}(A) = A_1 \times \cdots \times A_\rho$, where A_i are \mathbb{Z}_2 PI-reduced superalgebras, and $\text{ind}_{\mathbb{Z}_2}(A_i) = \text{ind}_{\mathbb{Z}_2}(A)$ for all $i = 1, \dots, \rho$. There exists $\hat{\mu} \in \mathbb{N}_0$ such that $S_\mu(A) = S_\mu(\mathcal{O}(A)) = \bigcup_{i=1}^\rho S_\mu(A_i) \subseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{Y}(A))$ holds for any $\mu \geq \hat{\mu}$.*

Proof. Suppose that A is \mathbb{Z}_2 PI-equivalent to a direct product $A_1 \times \cdots \times A_{\hat{\rho}}$ of \mathbb{Z}_2 PI-reduced superalgebras. By Lemma 9 $\text{ind}_{\mathbb{Z}_2}(A) = \max_{1 \leq i \leq \hat{\rho}} \text{ind}_{\mathbb{Z}_2}(A_i)$. Assume that the Kemer index has the maximal value for A_i with $i = 1, \dots, \rho$. Then the superalgebras $\mathcal{O}(A) = A_1 \times \cdots \times A_\rho$, $\mathcal{Y}(A) = A_{\rho+1} \times \cdots \times A_{\hat{\rho}}$ satisfy the assertion of the lemma. \square

Definition 11 $\mathcal{O}(A)$ is called the senior part of A , $\mathcal{Y}(A)$ is called the junior part of A . The algebras A_i ($i = 1, \dots, \rho$) are called the senior components of A . $\hat{\mu}(A)$ is the minimal $\hat{\mu} \in \mathbb{N}_0$ satisfying the assertion of Lemma 16.

The next lemma shows the relation between the Kemer index and the parameters of a \mathbb{Z}_2 PI-reduced superalgebra.

Lemma 17 *Given a \mathbb{Z}_2 PI-reduced superalgebra A we have $\beta(A) = \dim_{\mathbb{Z}_2} A$.*

Proof. For a nilpotent superalgebra the assertion is trivial. Consider any non-nilpotent \mathbb{Z}_2 PI-reduced algebra A with $\dim_{\mathbb{Z}_2} A = (t_0, t_1)$. By Lemma 5 it is enough to find a polynomial of the type $(t_0, t_1; 0; \hat{s})$ that is not a graded identity of A for any $\hat{s} \in \mathbb{N}$. Assume that A has the elementary decomposition (1). For any $l = 1, \dots, p$ consider the graded monomial $W_l(Y_{(l)}, Z_{(l)})$ of the type (5) or (7) constructing for the simple component C_l (see Lemma 13). $W_l(Y_{(l)}, Z_{(l)})$ depends on the disjoint sets of graded variables $Y_{(l)} = \bigcup_{d=1}^{\hat{s}} (Y_{(l),(d,\bar{0})} \cup Y_{(l),(d,\bar{1})})$, and $Z_{(l)} = \{z_{(l),(ij)}, | i, j = 1, \dots, s_l\}$. Where $Y_{(l)} = \{y_{(l),(d,(ij))} \in X^{\mathbb{Z}_2} \mid 1 \leq i, j \leq s_l; 1 \leq d \leq \hat{s}\}$ for $C_l = M_{k_l, m_l}(F)$, and $Y_{(l)} = \{y_{(l),(d,\theta,(ij))} \in X^{\mathbb{Z}_2} \mid \theta = \bar{0}, \bar{1}; 1 \leq i, j \leq s_l; 1 \leq d \leq \hat{s}\}$ for $C_l = M_{k_l}(F[c])$. We have that $Y_{(l),(d,\theta)} \subseteq X_\theta$, $Y_{d,\theta} = \bigcup_{l=1}^p Y_{(l),(d,\theta)}$, and $\deg_{\mathbb{Z}_2} z_{(l),(ij)} = \deg_{\mathbb{Z}_2} E_{lij}$ in C_l . It is clear that $|Y_{d,\theta}| = t_\theta$ for any $d = 1, \dots, \hat{s}$, $\theta \in \mathbb{Z}/2\mathbb{Z}$.

Then the appropriate elementary evaluation of the word $W_l(Y_{(l)}, Z_{(l)})$ in C_l (see Lemma 13) is equal to the non-zero element $E_{l11}\bar{c}_l$, where $\bar{c}_l = 1$ or $\bar{c}_l = c$. By Lemma 14 we can assume that A contains an element $\varepsilon_1 r_1 \varepsilon_2 \cdots \varepsilon_{p-1} r_{p-1} \varepsilon_p \neq 0$, where $r_l \in J$ are some $\mathbb{Z}/2\mathbb{Z}$ -homogeneous radical elements, and $\varepsilon_l = \sum_{i_l=1}^{s_l} E_{li_l i_l}$ is the unit of C_l ($l = 1, \dots, p$). Let us take p nontrivial graded polynomials

$$f_l(\tilde{X}_{(l)}, Y_{(l)}, Z_{(l)}) = \sum_{i_l=1}^{s_l} \tilde{x}_{l,(i_l)} W_l(Y_{(l)}, Z_{(l)}) \tilde{x}_{l,(i_l, \bar{c}_l)} \quad (9)$$

depending on the additional set of graded variables $\tilde{X}_{(l)} = \{\tilde{x}_{l,(i_l)}, \tilde{x}_{l,(i_l, \bar{c}_l)} \in X^{\mathbb{Z}_2} \mid 1 \leq i_l \leq s_l; \deg_{\mathbb{Z}_2} \tilde{x}_{l,(i_l)} = \deg_{\mathbb{Z}_2} E_{li_l 1}, \deg_{\mathbb{Z}_2} \tilde{x}_{l,(i_l, \bar{c}_l)} = \deg_{\mathbb{Z}_2} E_{li_l \bar{c}_l}\}$. Notice that f_l is not a multihomogeneous polynomial, although it is linear in $Y_{(l)}$, and $Z_{(l)}$. But f_l is homogeneous in the grading of even degree, since any monomial $\tilde{x}_{l,(i_l)} W_l(Y_{(l)}, Z_{(l)}) \tilde{x}_{l,(i_l, \bar{c}_l)}$ has even degree (it has the same $\mathbb{Z}/2\mathbb{Z}$ -degree as $E_{li_l i_l}$ in C_l).

Then the polynomial

$$f(\tilde{X}, Y, Z) = \left(\prod_{d=1}^{\hat{s}} \mathcal{A}_{Y_{d,\bar{0}}} \mathcal{A}_{Y_{d,\bar{1}}} \right) (f_1 x_1 f_2 x_2 \cdots x_{p-1} f_p) \quad (10)$$

is linear in $Y \cup Z = (\bigcup_{l=1}^p Y_{(l)}) \cup (\bigcup_{l=1}^p Z_{(l)})$, and alternating in any set $Y_{d,\theta} \subseteq X_\theta$, $\theta \in \mathbb{Z}/2\mathbb{Z}$, $d = 1, \dots, s$. It follows from Lemma 13 that the evaluation

$$\begin{aligned} y_{(l),(d,(i_l j_l))} &= \varepsilon_l E_{li_l j_l} \varepsilon_l, & y_{(l),(d,\bar{0},(i_l j_l))} &= \varepsilon_l E_{li_l j_l} \varepsilon_l, \\ y_{(l),(d,\bar{1},(i_l j_l))} &= \varepsilon_l (E_{li_l j_l} c) \varepsilon_l, & z_{(l),(i_l j_l)} &= \varepsilon_l E_{li_l j_l} \varepsilon_l, \\ \tilde{x}_{l,(i_l)} &= \varepsilon_l E_{li_l 1} \varepsilon_l, & \tilde{x}_{l,(i_l, \bar{c}_l)} &= \varepsilon_l (E_{li_l \bar{c}_l}) \varepsilon_l, \\ x_q &= \varepsilon_q r_q \varepsilon_{q+1}, \\ 1 \leq i_l, j_l &\leq s_l, \quad l = 1, \dots, p; \quad q = 1, \dots, p-1; \quad d = 1, \dots, \hat{s}, \end{aligned}$$

of the polynomial $f(\tilde{X}, Y, Z)$ is equal to $\varepsilon_1 r_1 \varepsilon_2 \cdots \varepsilon_{p-1} r_{p-1} \varepsilon_p \neq 0$.

Therefore we obtain $f \notin \text{Id}^{\mathbb{Z}_2}(A)$. Hence at least one multihomogeneous component \tilde{f} of f also is not a graded identity of A . \tilde{f} is alternating in any set $Y_{d,\theta}$ ($\theta \in \mathbb{Z}/2\mathbb{Z}$, $d = 1, \dots, \hat{s}$). Thus \tilde{f} is the required polynomial. \square

5 Exact polynomials.

Definition 12 *Given a finite dimensional superalgebra A with elementary decomposition an elementary evaluation $(a_1, \dots, a_n) \in A^n$ (namely, $a_i \in D \cup U \subseteq A$ (2), (3)) is called incomplete if there exists $j = 1, \dots, p$ such that*

$$\{a_1, \dots, a_n\} \cap (C_j \cup \{\varepsilon_j u \varepsilon_l, \varepsilon_l u \varepsilon_j \mid u \in U, l = 1, \dots, p+1\}) = \emptyset.$$

Otherwise (a_1, \dots, a_n) is called complete.

Definition 13 *An elementary evaluation $(a_1, \dots, a_n) \in A^n$ is called thin if it contains less than $\text{nd}(A) - 1$ radical elements (not necessarily distinct).*

Definition 14 *We say that a multilinear graded polynomial $f(x_1, \dots, x_n) \in F\langle X^{\mathbb{Z}_2} \rangle$ is exact for a finite dimensional superalgebra A with elementary decomposition if $f(a_1, \dots, a_n) = 0$ holds in A for any thin or incomplete evaluation $(a_1, \dots, a_n) \in A^n$.*

Lemma 18 *If A is a \mathbb{Z}_2 PI-reduced superalgebra then any multilinear polynomial of the type $(\text{dims}_{\mathbb{Z}_2} A; \text{nd}(A) - 1; 0)$ is exact for A .*

Proof. It is clear that a \mathbb{Z}_2 PI-reduced algebra either is not semisimple or is $\mathbb{Z}/2\mathbb{Z}$ -simple. For a $\mathbb{Z}/2\mathbb{Z}$ -graded simple finite dimensional algebra any multilinear graded polynomial can be assumed exact. A multilinear polynomial of the type $(0, 0; \text{nd}(A) - 1; 0)$ has degree greater or equal to $(\text{nd}(A) - 1)$. Hence it is assumed to be exact for a nilpotent algebra A .

Suppose that A is not nilpotent, and f is a multilinear polynomial of the type $(\text{dims}_{\mathbb{Z}_2} A; \text{nd}(A) - 1; 0)$. In a thin evaluation at least one of $\tilde{s} = \text{nd}(A) - 1$ collections of τ_j -alternating variables of f will be completely replaced by semisimple elements. Since $\tau_j > \text{dims}_{\mathbb{Z}_2} A$ for any $j = 1, \dots, \tilde{s}$ then the result of such evaluation will be zero.

Any simple $\mathbb{Z}/2\mathbb{Z}$ -graded component C_l of A (Lemma 1) has the unit $\varepsilon_l \in C_{l\bar{0}}$. Hence $\dim_F C_{l\bar{0}} > 0$ for any $l = 1, \dots, p$. Therefore an incomplete evaluation can not contain all semisimple elements of the even degree from the base D (2). Taking into account the conditions $\tau_j > \text{dims}_{\mathbb{Z}_2} A$ for any $j = 1, \dots, \tilde{s}$ we obtain that at least two variables of every collection of τ_j -alternating variables of f must be substituted by radical elements, otherwise the result of the substitution will be zero. Thus in any case the result of an incomplete evaluation of the polynomial f is zero. \square

Lemma 19 *Any nonzero \mathbb{Z}_2 PI-reduced superalgebra A has an exact polynomial, that is not a graded identity of A .*

Proof. For a nilpotent superalgebra A the assertion follows from Lemma 18. Suppose that A is a non-nilpotent superalgebra with the decomposition (1). Consider its subalgebras $A_i = (\prod_{\substack{1 \leq j \leq p \\ j \neq i}} C_j) \oplus J(A)$, $i = 1, \dots, p$. Take $q = \dim_F J(A)$,

$s = \text{nd}(A) - 1$. Then by Lemma 2 $\tilde{A} = A_1 \times \dots \times A_p \times \mathcal{R}_{q,s}(A)$ is a finite dimensional superalgebra with elementary decomposition satisfying $\text{Id}^{\mathbb{Z}_2}(A) \subseteq \text{Id}^{\mathbb{Z}_2}(\tilde{A})$. Let $\{r_1, \dots, r_q\} \subseteq U$ be a homogeneous basis of $J(A)$ (3), $\deg_{\mathbb{Z}_2} r_i = \theta_i$ ($i = 1, \dots, q$). Consider the map φ defined by the next equalities $\varphi(x_{i\theta_i}) = r_i$ for $i = 1, \dots, q$, and $\varphi(b) = b$ for any $b \in B$. φ can be extended to a surjective graded homomorphism $\varphi : B(X_q^{\mathbb{Z}_2}) \rightarrow A$. It follows that any multilinear polynomial $f \in \text{Id}^{\mathbb{Z}_2}(\mathcal{R}_{q,s}(A))$ is turned into zero under any thin substitution. It is also clear that any incomplete substitution in a multilinear graded polynomial $f \in \text{Id}^{\mathbb{Z}_2}(\times_{i=1}^p A_i)$ yields zero. Therefore, any multilinear polynomial $f \in \text{Id}^{\mathbb{Z}_2}(\tilde{A})$ is exact for A . Remark that $\text{cpar}_{\mathbb{Z}_2}(A_i) < \text{cpar}_{\mathbb{Z}_2}(A)$ ($1 \leq i \leq p$), and $\text{cpar}_{\mathbb{Z}_2}(\mathcal{R}_{q,s}(A)) < \text{cpar}_{\mathbb{Z}_2}(A)$. Since A is \mathbb{Z}_2 PI-reduced then $\text{Id}^{\mathbb{Z}_2}(A) \subsetneq \text{Id}^{\mathbb{Z}_2}(\tilde{A})$. Any multilinear graded polynomial f such that $f \in \text{Id}^{\mathbb{Z}_2}(\tilde{A})$, and $f \notin \text{Id}^{\mathbb{Z}_2}(A)$ satisfies the assertion of the lemma. \square

Lemma 20 *Let A be a finite dimensional superalgebra with an elementary decomposition, h an exact polynomial for A , and $\bar{a} \in A^n$ is any complete evaluation of h containing $\tilde{s} = \text{nd}(A) - 1$ radical elements. Then for any $\mu \in \mathbb{N}_0$ there exist a graded polynomial $h_\mu \in \mathbb{Z}_2 T[h]$, and an elementary evaluation \bar{u} of h_μ by elements of A such that:*

1. $h_\mu(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\mu}, \tilde{X}, \tilde{Z})$ is τ_j -alternating in any set \tilde{Y}_j with $\tau_j > \beta = \dim_{\mathbb{Z}_2} A$ for all $j = 1, \dots, \tilde{s}$, and is β -alternating in any \tilde{Y}_j for $j = \tilde{s} + 1, \dots, \tilde{s} + \mu$ (all the sets $\tilde{Y}_j, \tilde{X}, \tilde{Z}$ are disjoint),
2. $h_\mu(\bar{u}) = h(\bar{a})$,
3. all the variables of $\tilde{X} \cup \tilde{Z}$ are replaced by semisimple elements.

Proof. If $h(\bar{a}) = 0$ then the assertion of lemma is trivial. It is sufficient to take any consequence $h_\mu(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\mu}) \in \mathbb{Z}_2 T[h]$ that is alternating in all \tilde{Y}_j as required (we assume here that $\tilde{X} \cup \tilde{Z} = \emptyset$), and replace the variables of the alternating sets \tilde{Y}_j by equal elements. Particularly, from conditions $\text{nd}(A) = 1$, $p \geq 2$ it follows that $h(\bar{a}) = 0$.

Assume that $h(\bar{a}) \neq 0$. Consider the case $\text{nd}(A) > 1$, $p \geq 2$ in the decomposition (1) of the superalgebra A . We can assume for simplicity that the evaluation \bar{a} has the form $\bar{a} = (\varepsilon_1 r_1 \varepsilon_{l'_1}, \varepsilon_{l'_2} r_2 \varepsilon_2, \dots, \varepsilon_{p-1} r_p \varepsilon_p, \varepsilon_{l'_{q+1}} r_{q+1} \varepsilon_{l''_{q+1}}, \dots, \varepsilon_{l'_s} r_s \varepsilon_{l''_s}, b_1, \dots, b_{n-\tilde{s}})$, where $1 \leq l'_s, l''_s \leq p+1$, $\{1, \dots, p\} \subseteq \{l'_s, l''_s \mid 1 \leq s \leq q\}$ for some $q \leq \tilde{s}$, $l'_s \neq l''_s$ for any $1 \leq s \leq q$. Namely, in a complete evaluation all minimal orthogonal idempotents $\varepsilon_1, \dots, \varepsilon_p$ appear in mixed radical elements (in elements of the type $\varepsilon_{l'} r_l \varepsilon_{l''} \in U$ (3) with $l' \neq l''$). We assume that they appear in the first q mixed radical elements.

The last elements $n - \tilde{s}$ of the evaluation $\bar{a} \ b_1, \dots, b_{n-\tilde{s}} \in D \ (2)$ are supposed to be semisimple, and the first \tilde{s} elements $\varepsilon_{l'_s} r_s \varepsilon_{l'_s} \in U \ (3)$ are radical.

Let us take for any $l = 1, \dots, p$ the polynomial $f_l(\tilde{X}_{(l)}, Y_{(l)}, Z_{(l)})$ defined by (9) in Lemma 17 assuming $\hat{s} = \tilde{s} + \mu = \text{nd}(A) - 1 + \mu$. Here $Y_{(l)} = \bigcup_{d=1}^{\tilde{s}+\mu} (Y_{(l),(d,\bar{0})} \cup Y_{(l),(d,\bar{1})})$, and $Y_{d,\theta} = \bigcup_{l=1}^p (Y_{(l),(d,\bar{0})} \cup Y_{(l),(d,\bar{1})}) \ (d = 1, \dots, \tilde{s} + \mu, \ \theta \in \mathbb{Z}/2\mathbb{Z})$. consider also a new set of graded variables $\tilde{y}_s \in X^{\mathbb{Z}_2}$ such that $\deg_{\mathbb{Z}_2} \tilde{y}_s = \deg_{\mathbb{Z}_2} r_s, \ 1 \leq s \leq \tilde{s}$. Let us denote $\tilde{Y}_{d,\theta} = Y_{d,\theta} \cup \{\tilde{y}_d\}$ if $\deg_{\mathbb{Z}_2} \tilde{y}_d = \theta$, and $\tilde{Y}_{d,\theta} = Y_{d,\theta}$ otherwise $d = 1, \dots, \mu + \tilde{s}$.

Then we obtain $\bar{\tau}_d = (\tau_{d\bar{0}}, \tau_{d\bar{1}}) = (|\tilde{Y}_{d,\bar{0}}|, |\tilde{Y}_{d,\bar{1}}|) > (|Y_{d,\bar{0}}|, |Y_{d,\bar{1}}|)$ if $d = 1, \dots, \tilde{s}$. And $\bar{\tau}_d = (|\tilde{Y}_{d,\bar{0}}|, |\tilde{Y}_{d,\bar{1}}|) = (|Y_{d,\bar{0}}|, |Y_{d,\bar{1}}|)$ for any $d = \tilde{s} + 1, \dots, \tilde{s} + \mu$. Here $(|Y_{d,\bar{0}}|, |Y_{d,\bar{1}}|) = \text{dims}_{\mathbb{Z}_2} A$ for any d .

Denote by $\tilde{Y}_d = \tilde{Y}_{d,\bar{0}} \cup \tilde{Y}_{d,\bar{1}} \ (d = 1, \dots, \tilde{s} + \mu)$; $\tilde{X} = (\bigcup_{l=1}^p \tilde{X}_l) \cup \{x_1, \dots, x_{n-\tilde{s}}\}$; $\tilde{Z} = \bigcup_{l=1}^p Z_l$. Consider the polynomials

$$\begin{aligned} h'(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\mu}, \tilde{X}, \tilde{Z}) &= h\left(f_1(\tilde{X}_{(1)}, Y_{(1)}, Z_{(1)})\tilde{y}_1, \tilde{y}_2 f_2(\tilde{X}_{(2)}, Y_{(2)}, Z_{(2)}), \dots, \right. \\ &\quad \left. f_{p-1}(\tilde{X}_{(p-1)}, Y_{(p-1)}, Z_{(p-1)})\tilde{y}_q f_p(\tilde{X}_{(p)}, Y_{(p)}, Z_{(p)}), \tilde{y}_{q+1}, \dots, \tilde{y}_{\tilde{s}}, x_1, \dots, x_{n-\tilde{s}}\right); \\ h_\mu(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\mu}, \tilde{X}, \tilde{Z}) &= \left(\prod_{d=1}^{\tilde{s}+\mu} (\mathcal{A}_{\tilde{Y}_{d,\bar{0}}} \mathcal{A}_{\tilde{Y}_{d,\bar{1}}})\right) h'(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\mu}, \tilde{X}, \tilde{Z}). \end{aligned}$$

The polynomial h_μ is linear in variables $\tilde{Y} = \bigcup_{d=1}^{\tilde{s}+\mu} \tilde{Y}_d$, and $\bar{\tau}_d$ -alternating in any \tilde{Y}_d ($d = 1, \dots, \tilde{s} + \mu$). Although h_μ is not multilinear and is not multihomogeneous in general.

Consider the following evaluation of the polynomial h_μ in the superalgebra A

$$\begin{aligned} y_{(l),(d,(i_l j_l))} &= \varepsilon_l E_{l i_l j_l} \varepsilon_l, & y_{(l),(d,\bar{0},(i_l j_l))} &= \varepsilon_l E_{l i_l j_l} \varepsilon_l, \\ z_{(l),(d,\bar{1},(i_l j_l))} &= \varepsilon_l (E_{l i_l j_l} c) \varepsilon_l, & z_{(l),(i_l j_l)} &= \varepsilon_l E_{l i_l j_l} \varepsilon_l, \\ \tilde{x}_{l,(i_l)} &= \varepsilon_l E_{l i_l 1} \varepsilon_l, & \tilde{\tilde{x}}_{l,(j_l, \bar{c}_l)} &= \varepsilon_l (E_{l 1 j_l} \bar{c}_l) \varepsilon_l, \\ \tilde{y}_s &= a_s = \varepsilon_{l'_s} r_s \varepsilon_{l'_s}; & x_{n'} &= a_{n'+\tilde{s}} = b_{n'}, \\ l &= 1, \dots, p; & 1 \leq i_l, j_l &\leq s_l, \\ d &= 1, \dots, \tilde{s} + \mu; & 1 \leq s \leq \tilde{s}; & \quad 1 \leq n' \leq n - \tilde{s}. \end{aligned} \tag{11}$$

The evaluations of the variables y, z, \tilde{x} , and $\tilde{\tilde{x}}$ are defined as in Lemma 17. It is clear that (11) is an elementary evaluation and satisfies the third claim of the lemma.

Due to the evaluation of the variables z, \tilde{x} , and $\tilde{\tilde{x}}$, the polynomial f_l can contain only elements of the simple component C_l or elements $\varepsilon_l r_s \varepsilon_l$, otherwise, we get zero. The second case will give us a thin evaluation of the polynomial h , thus, such summands are also zero. Therefore, the evaluation (11) of the polynomial h_μ gives the same result as this evaluation of the polynomial

$$h(f'_1 \tilde{y}_1, \tilde{y}_2 f'_2, \dots, f'_{p-1} \tilde{y}_q f'_p, \tilde{y}_{q+1}, \dots, \tilde{y}_{\tilde{s}}, x_1, \dots, x_{n-\tilde{s}}),$$

where $f'_l = (\prod_{d=1}^{\tilde{s}+\mu} \mathcal{A}_{\tilde{Y}_{d,\bar{0}}} \mathcal{A}_{\tilde{Y}_{d,\bar{1}}}) f_l$. Similarly to arguments of Lemmas 17, 13, we can see that the result of our evaluation of f'_l is equal to ε_l . Thus (11) for h_μ gives the

result $h(\varepsilon_1 r_1 \varepsilon_{l'_1}, \varepsilon_{l'_2} r_2 \varepsilon_2, \dots, \varepsilon_{p-1} r_p \varepsilon_p, \varepsilon_{l'_{q+1}} r_{q+1} \varepsilon_{l''_{q+1}}, \dots, \varepsilon_{l'_s} r_s \varepsilon_{l''_s}, b_1, \dots, b_{n-\tilde{s}}) = h(a_1, \dots, a_n)$. Therefore h_μ is the desired polynomial. The evaluation (11) can be taken as \bar{u} .

Consider the case $\text{nd}(A) \geq 1$, and $p = 1$. If $\varepsilon_1 h(a_1, \dots, a_n) = 0$ then among \tilde{s} radical elements of \bar{a} there is a homogeneous in the grading radical element of the type $\varepsilon_2 r \varepsilon_1 \in \varepsilon_2 J(A) \varepsilon_1$, where ε_2 is the adjoint idempotent of A (since the substitution \bar{a} is complete). We can suppose without loss of generality that $a_1 = \varepsilon_2 r_1 \varepsilon_1$, and the first $\tilde{s} = (\text{nd}(A) - 1)$ elements of \bar{a} are radical $a_2 = \varepsilon_{l'_2} r_2 \varepsilon_{l''_2}, \dots, a_{\tilde{s}} = \varepsilon_{l'_s} r_s \varepsilon_{l''_s} \in J(A)$, $a_{n'+\tilde{s}} = b_{n'} \in D$, for $n' = 1, \dots, n - \tilde{s}$. Then using the same arguments as in the previous case $p \geq 2$, assuming that $p = 1$, $q = 1$, $l'_s, l''_s \in \{1, 2\}$, we can prove that the evaluation (11) of the polynomial

$$h_\mu = \left(\prod_{d=1}^{\tilde{s}+\mu} \mathcal{A}_{\tilde{Y}_{d,\bar{0}}} \mathcal{A}_{\tilde{Y}_{d,\bar{1}}} \right) h\left(\tilde{y}_1 \cdot f_1(\tilde{X}_{(1)}, Y_{(1)}, Z_{(1)}), \tilde{y}_2, \dots, \tilde{y}_{\tilde{s}}, x_1, \dots, x_{n-\tilde{s}}\right)$$

is equal to $h(a_1, \dots, a_n)$.

If $\varepsilon_1 h(a_1, \dots, a_n) \neq 0$ then the similar arguments as in the case $p \geq 2$ show that the result of the evaluation (11) of the polynomial

$$h_\mu = \left(\prod_{d=1}^{\tilde{s}+\mu} (\mathcal{A}_{\tilde{Y}_{d,\bar{0}}} \mathcal{A}_{\tilde{Y}_{d,\bar{1}}}) \right) f_1(\tilde{X}_{(1)}, Y_{(1)}, Z_{(1)}) \cdot h(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\tilde{s}}, x_1, \dots, x_{n-\tilde{s}})$$

is equal to $\varepsilon_1 h(a_1, \dots, a_n) = h(a_1, \dots, a_n)$. Here variables \tilde{y}_s can not take places inside f_1 with a nonzero result, since it will give a thin evaluation of h .

In both of the last cases the polynomial h_μ and the corresponding evaluation possess all desired properties.

The case $p = 0$ is trivial. In this case the superalgebra A is nilpotent, and $\text{dms}_{\mathbb{Z}_2} A = (0, 0)$. Since a multihomogeneous graded polynomial h is exact for A and is not a graded identity of A ($h(\bar{a}) \neq 0$) then the full degree of h is $(\text{nd}(A) - 1)$, and h has the type $(0, 0; \text{nd}(A) - 1; \mu) = (\text{dms}_G A; \text{nd}(A) - 1; \mu)$ for any $\mu \in \mathbb{N}_0$. Thus in this case $h_\mu = h$, $\bar{u} = \bar{a}$. \square

Lemma 20 has the following important corollaries.

Lemma 21 *Let A be a \mathbb{Z}_2 PI-reduced algebra then $\text{ind}_{\mathbb{Z}_2}(A) = \text{par}_{\mathbb{Z}_2}(A)$. If f is an exact polynomial for A , and $f \notin \text{Id}^{\mathbb{Z}_2}(A)$ then $\mathbb{Z}_2 T[f] \cap S_\mu(A) \neq \emptyset$ for any $\mu \in \mathbb{N}_0$.*

Proof. By Lemma 19 A has an exact polynomial $\tilde{f} \notin \text{Id}^{\mathbb{Z}_2}(A)$. \tilde{f} can be nonzero only for a complete evaluation containing exactly $(\text{nd}(A) - 1)$ radical elements. Lemma 20 implies that \tilde{f} has a nontrivial consequence $\tilde{g} \notin \text{Id}^{\mathbb{Z}_2}(A)$ of the type $(\text{dms}_{\mathbb{Z}_2} A; \text{nd}(A) - 1; \mu)$ for any $\mu \in \mathbb{N}_0$. By Lemma 17 we have $\beta(A) = \text{dms}_{\mathbb{Z}_2} A$. Then Definition 7 implies $\gamma(A) > \text{nd}(A) - 1$. Taking into account that $\text{ind}_{\mathbb{Z}_2}(A) \leq \text{par}_{\mathbb{Z}_2}(A)$ we obtain $\gamma(A) = \text{nd}(A)$.

Moreover by Lemma 20 any exact for A polynomial f such that $f \notin \text{Id}^{\mathbb{Z}_2}(A)$ has a nontrivial consequence $g_\mu \in \mathbb{Z}_2 T[f]$ for any $\mu \in \mathbb{N}_0$. Where g_μ is μ -boundary polynomial for A . \square

Lemma 21 along with Lemma 18 immediately implies

Lemma 22 Any multilinear μ -boundary polynomial for a \mathbb{Z}_2 PI-reduced algebra A is exact for A .

Lemma 23 Given a \mathbb{Z}_2 PI-reduced algebra A , and an integer $\mu \in \mathbb{N}_0$ let $S_{A,\mu}$ be any set of graded polynomials of the type $(\beta(A); \gamma(A) - 1; \mu)$. Then if a multilinear polynomial $f \in \mathbb{Z}_2 T[S_{A,\mu}] + \text{Id}^{\mathbb{Z}_2}(A)$ then f is exact for A .

Proof. A multilinear graded polynomial $f \in \mathbb{Z}_2 T[S_{A,\mu}] + \text{Id}^{\mathbb{Z}_2}(A)$ has the form

$$f(x_1, \dots, x_n) = \sum_{\substack{1 \leq i \leq d_1 \\ 1 \leq j \leq d_2}} \alpha_{ij} v_i \cdot g_j(h_{i1}, \dots, h_{in_j}) \cdot w_i + g(x_1, \dots, x_n),$$

where $h_{i1}, \dots, h_{in_j}, v_i, w_i \in F\langle X^{\mathbb{Z}_2} \rangle$ are graded monomials, v_i, w_i are possibly empty; $g_j(z_{j1}, \dots, z_{jn_j}) \in F\langle X^{\mathbb{Z}_2} \rangle$ are full linearizations of some polynomials from the set $S_{A,\mu}$; g is a multilinear graded identity of A ; $\alpha_{ij} \in F$. It is clear that the multilinear graded polynomials $g_j(z_{j1}, \dots, z_{jn_j})$ have the type $(\beta(A); \gamma(A) - 1; \mu)$ ($\forall j = 1, \dots, d_2$), and the graded monomials $v_i \cdot h_{i1} \cdots h_{in_j} \cdot w_i$ are multilinear and depends on the same variables x_1, \dots, x_n as f for any i . Particularly, the monomials $h_{i1}(x_{\delta_1}, \dots, x_{\delta_{s_{i1}}}), \dots, h_{in_j}(x_{\lambda_1}, \dots, x_{\lambda_{s_{in_j}}})$ are multilinear, here we have $\{\delta_1, \dots, \delta_{s_{i1}}, \dots, \lambda_1, \dots, \lambda_{s_{in_j}}\} \subseteq \{1, \dots, n\}$.

Fix any $i = 1, \dots, d_1, j = 1, \dots, d_2$. Given an elementary evaluation (a_1, \dots, a_n) of f in A consider homogeneous elements of A , that are the evaluation of the monomials $h_{il} \quad \tilde{a}_{i1} = h_{i1}(a_{\delta_1}, \dots, a_{\delta_{s_{i1}}}), \dots, \tilde{a}_{in_j} = h_{in_j}(a_{\lambda_1}, \dots, a_{\lambda_{s_{in_j}}})$. Then \tilde{a}_{il} is either a semisimple element of A or radical, and

$$\begin{aligned} \tilde{a}_{il} &= \sum_{t_l=1}^{\dim \mathcal{R}_l} \alpha_{(il),t_l} u_{t_l}, & \alpha_{(il),t_l} &\in F, \quad \deg_{\mathbb{Z}_2} u_{t_l} = \deg_{\mathbb{Z}_2} \tilde{a}_{il} = \theta_l, \quad \forall t_l; \\ u_{t_l} &\in D(2), \quad \forall t_l, \quad \mathcal{R}_l = B_{\theta_l}, & \text{or} & \quad u_{t_l} \in U(3), \quad \forall t_l, \quad \mathcal{R}_l = J(A)_{\theta_l}. \end{aligned}$$

Then the evaluation (a_1, \dots, a_n) of $g_j(h_{i1}, \dots, h_{in_j})$ yields

$$\begin{aligned} g_j(\tilde{a}_{i1}, \dots, \tilde{a}_{in_j}) &= g_j\left(\sum_{t_1=1}^{\dim \mathcal{R}_1} \alpha_{(i1),t_1} u_{t_1}, \dots, \sum_{t_{n_j}=1}^{\dim \mathcal{R}_{n_j}} \alpha_{(in_j),t_{n_j}} u_{t_{n_j}}\right) = \\ &= \sum_{t_1, \dots, t_{n_j}} \alpha_{(i1),t_1} \cdots \alpha_{(in_j),t_{n_j}} g_j(u_{t_1}, \dots, u_{t_{n_j}}). \end{aligned}$$

All evaluations $(u_{t_1}, \dots, u_{t_{n_j}})$ of the polynomials g_j are elementary. The numbers of radical elements in $(u_{t_1}, \dots, u_{t_{n_j}})$ is equal to the number of radical elements in $(\tilde{a}_{i1}, \dots, \tilde{a}_{in_j})$ for all t_1, \dots, t_{n_j} , and does not exceed the number of radical elements in the initial evaluation (a_1, \dots, a_n) . Thus if (a_1, \dots, a_n) is a thin evaluation then $(u_{t_1}, \dots, u_{t_{n_j}})$ is also thin for all t_1, \dots, t_{n_j} and i, j .

$(\prod_{\substack{1 \leq l \leq p \\ l \neq k}} C_l) \oplus (\sum_{\substack{1 \leq l', l'' \leq p+1 \\ l' \neq k, l'' \neq k}} \varepsilon_{l'} J(A) \varepsilon_{l''})$ is a graded subalgebra of A for any k . There-

fore, if (a_1, \dots, a_n) is incomplete then $(\tilde{a}_{i1}, \dots, \tilde{a}_{in_j})$, and $(u_{t_1}, \dots, u_{t_{n_j}})$ are also incomplete. The last evaluations do not contain elements of $C_k \cup \{\varepsilon_k r \varepsilon_l, \varepsilon_l r \varepsilon_k \mid r \in U, 1 \leq l \leq p+1\}$ if elements of this set does not appear in (a_1, \dots, a_n) .

A is \mathbb{Z}_2 PI-reduced, and by Lemma 21 $\text{ind}_{\mathbb{Z}_2}(A) = \text{par}_{\mathbb{Z}_2}(A)$. Thus g_j is a multilinear graded polynomial of the type $(\beta(A); \gamma(A) - 1; \mu) = (\text{dims}_{\mathbb{Z}_2} A; \text{nd}(A) - 1; \mu)$, and by Lemma 18 g_j is exact for A (for any j).

Thus for any thin or incomplete evaluation (a_1, \dots, a_n) we obtain

$$g_j(\tilde{a}_{i1}, \dots, \tilde{a}_{in_j}) = \sum_{t_1, \dots, t_{n_j}} \alpha_{(il), t_1} \cdots \alpha_{(il), t_{n_j}} g_j(u_{t_1}, \dots, u_{t_{n_j}}) = 0.$$

Hence $f(a_1, \dots, a_n) = 0$, and f is exact for A . \square

Lemma 24 *Let Γ be a proper $\mathbb{Z}_2 T$ -ideal, and A be a \mathbb{Z}_2 PI-reduced algebra such that $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \text{ind}_{\mathbb{Z}_2}(A)$. Suppose that a graded polynomial f satisfies the conditions $f \notin \text{Id}^{\mathbb{Z}_2}(A)$, and $f \in K_{\hat{\mu}}(\Gamma) + \text{Id}^{\mathbb{Z}_2}(A)$ for some $\hat{\mu} \in \mathbb{N}_0$. Then $\mathbb{Z}_2 T[f] \cap S_{\mu}(A) \neq \emptyset$ holds for any $\mu \in \mathbb{N}_0$.*

Proof. The full linearization \tilde{f} of some multihomogeneous component of f also satisfies $\tilde{f} \in K_{\hat{\mu}}(\Gamma) + \text{Id}^{\mathbb{Z}_2}(A)$, $\tilde{f} \notin \text{Id}^{\mathbb{Z}_2}(A)$. Then by Lemma 23 \tilde{f} is exact for A . And by Lemma 21 we obtain that $\emptyset \neq (\mathbb{Z}_2 T[\tilde{f}] \cap S_{\mu}(A)) \subseteq (\mathbb{Z}_2 T[f] \cap S_{\mu}(A))$ for any $\mu \in \mathbb{N}_0$. \square

6 Representable graded algebras.

Let R be a commutative associative unitary F -algebra. Suppose that a $\mathbb{Z}/2\mathbb{Z}$ -graded F -algebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ has a structure of R -algebra satisfying $RA_{\theta} \subseteq A_{\theta}$, $\forall \theta \in \mathbb{Z}/2\mathbb{Z}$.

Definition 15 *Any R -linear mapping $\text{tr} : A_{\bar{0}} \rightarrow R$ is called trace on the $\mathbb{Z}/2\mathbb{Z}$ -graded R -algebra A .*

Observe that a trace on A is not necessary symmetric (not necessary satisfies $\text{tr}(ab) = \text{tr}(ba)$).

Denote by \mathcal{S} the free associative commutative unitary algebra generated by all symbols $\text{tr}(u)$ for nonempty associative noncommutative even monomials $u \in (F\langle X^{\mathbb{Z}_2} \rangle)_{\bar{0}}$ over $X^{\mathbb{Z}_2}$. We say that $FS\langle X^{\mathbb{Z}_2} \rangle = F\langle X^{\mathbb{Z}_2} \rangle \otimes_F \mathcal{S}$ is free $\mathbb{Z}/2\mathbb{Z}$ -graded algebra with trace. We assume that $(f \otimes s_1)s_2 = s_2(f \otimes s_1) = f \otimes (s_1 s_2)$ for all $f \in F\langle X \rangle$, $s_1, s_2 \in \mathcal{S}$. The $\mathbb{Z}/2\mathbb{Z}$ -grading on $FS\langle X^{\mathbb{Z}_2} \rangle$ is induced from $F\langle X^{\mathbb{Z}_2} \rangle$ (\mathcal{S} is supposed to be trivially graded). Elements of $FS\langle X^{\mathbb{Z}_2} \rangle$ are called graded polynomials with trace. Elements of \mathcal{S} are called pure trace polynomials.

We identify $F\langle X^{\mathbb{Z}_2} \rangle \otimes_F 1$ with $F\langle X^{\mathbb{Z}_2} \rangle$. The symbol \otimes usually is omitted in the notation of graded polynomials with trace. The concept of degrees (homogeneity in the sense of degree, multilinearity, alternating, etc.) of graded polynomials with trace or pure trace polynomials is defined in similar way to ordinary graded polynomials assuming $\deg_x \text{tr}(u) = \deg_x u$ for any $x \in X^{\mathbb{Z}_2}$.

The \mathcal{S} -linear function of trace $\text{tr} : (FS\langle X^{\mathbb{Z}_2} \rangle)_e \rightarrow \mathcal{S}$ is defined on the \mathcal{S} -algebra $FS\langle X^{\mathbb{Z}_2} \rangle$ by the formula $\text{tr}(\sum_i u_i s_i) = \sum_i \text{tr}(u_i) s_i$, where $u_i \in (F\langle X^{\mathbb{Z}_2} \rangle)_e$ are monomials on $X^{\mathbb{Z}_2}$ of the even graded degree, $s_i \in \mathcal{S}$.

Let A be a $\mathbb{Z}/2\mathbb{Z}$ -graded R -algebra with trace, $f(x_1, \dots, x_n) \in FS\langle X^{\mathbb{Z}_2} \rangle$ be a graded polynomial with trace. A satisfies the graded identity with trace $f = 0$ if $f(a_1, \dots, a_n) = 0$ holds in A for any $a_1, \dots, a_n \in A$. The ideal of graded identities with trace of A $\text{SI}^{\mathbb{Z}_2}(A) = \{f \in FS\langle X^{\mathbb{Z}_2} \rangle \mid A \text{ satisfies } f = 0\}$ is $\mathbb{Z}/2\mathbb{Z}$ -graded \mathcal{S} -ideal of $FS\langle X^{\mathbb{Z}_2} \rangle$ closed under all graded endomorphisms of the algebra $FS\langle X^{\mathbb{Z}_2} \rangle$, and preserving the trace. $\text{SI}^{\mathbb{Z}_2}(A)$ also satisfies the condition $g \cdot \text{tr}(f) \in \text{SI}^{\mathbb{Z}_2}(A)$, for any $g \in FS\langle X^{\mathbb{Z}_2} \rangle$, and for any $f \in (\text{SI}^{\mathbb{Z}_2}(A))_{\bar{0}}$. Ideals of $FS\langle X^{\mathbb{Z}_2} \rangle$ with these properties are called \mathbb{Z}_2 TS-ideals. Given a \mathbb{Z}_2 TS-ideal $\tilde{\Gamma}$, and polynomials with trace $f, g \in FS\langle X^{\mathbb{Z}_2} \rangle$ we write $f = g \pmod{\tilde{\Gamma}}$ if $f - g \in \tilde{\Gamma}$. $\mathbb{Z}_2TS[\mathcal{V}]$ is the \mathbb{Z}_2 TS-ideal generated by a set $\mathcal{V} \subseteq FS\langle X^{\mathbb{Z}_2} \rangle$.

Let $\tilde{\Gamma} \trianglelefteq FS\langle X^{\mathbb{Z}_2} \rangle$ be a \mathbb{Z}_2 TS-ideal of $FS\langle X^{\mathbb{Z}_2} \rangle$. Denote by $I = \text{Span}_F\{\text{tr}(f)v \mid f \in \tilde{\Gamma}_{\bar{0}}, v \in \mathcal{S}\} \trianglelefteq \mathcal{S}$ the ideal of \mathcal{S} generated by all elements of the form $\text{tr}(f)$ for all polynomials with trace $f \in \tilde{\Gamma}$ of the even graded degree. Let $\bar{\mathcal{S}} = \mathcal{S}/I$ be the quotient algebra. Then the quotient algebra $\overline{FS}\langle X^{\mathbb{Z}_2} \rangle = FS\langle X^{\mathbb{Z}_2} \rangle / \tilde{\Gamma}$ is $\mathbb{Z}/2\mathbb{Z}$ -graded and has the well-defined structure of $\bar{\mathcal{S}}$ -algebra. $\bar{\mathcal{S}}$ -linear function $\text{tr} : (FS\langle X^{\mathbb{Z}_2} \rangle / \tilde{\Gamma})_{\bar{0}} \rightarrow \bar{\mathcal{S}}$ is naturally defined by the equalities $\text{tr}(a + \tilde{\Gamma}) = \text{tr}(a) + I$ for any $a \in (FS\langle X^{\mathbb{Z}_2} \rangle)_{\bar{0}}$. The ideal of graded identities with trace of $\overline{FS}\langle X^{\mathbb{Z}_2} \rangle$ coincides with $\tilde{\Gamma}$. Moreover $FS\langle X^{\mathbb{Z}_2} \rangle / \tilde{\Gamma}$ is the relatively free $\mathbb{Z}/2\mathbb{Z}$ -graded algebra with trace for the \mathbb{Z}_2 TS-ideal $\tilde{\Gamma}$.

We also can consider the free associative $\mathbb{Z}/2\mathbb{Z}$ -graded algebra with trace $FS\langle X_{\nu}^{\mathbb{Z}_2} \rangle$ and the relatively free $\mathbb{Z}/2\mathbb{Z}$ -graded algebras with trace $FS\langle X_{\nu}^{\mathbb{Z}_2} \rangle / (\tilde{\Gamma} \cap FS\langle X_{\nu}^{\mathbb{Z}_2} \rangle)$ of a finite rank $\nu \in \mathbb{N}$.

Let us take a finite dimensional F -superalgebra $A = B \oplus J$ with the Jacobson radical $J = J(A)$, and the semisimple part B . Then the even component of the semisimple part $B_{\bar{0}} = B \cap A_{\bar{0}}$ is a finite dimensional subalgebra of A with $\dim_F B_{\bar{0}} = t_{\bar{0}}$, where $\dim_{\mathbb{Z}_2} A = (t_{\bar{0}}, t_{\bar{1}})$. Since the left regular representation $\mathfrak{T} : B_{\bar{0}} \rightarrow M_{t_1}(F)$ of $B_{\bar{0}}$ is injective then $B_{\bar{0}}$ is isomorphic to a subalgebra of the matrix algebra $M_{t_0}(F)$.

Therefore the trace $\text{tr} : A_{\bar{0}} \rightarrow F$ on A is well defined by the rule

$$\text{tr}(a) = \text{tr}(b + r) = \text{Tr}(\mathfrak{T}(b)), \quad a \in A_{\bar{0}}, \quad b \in B_{\bar{0}}, \quad r \in J_{\bar{0}}, \quad (12)$$

where Tr is the usual trace of a linear operator.

Lemma 25 *A finite dimensional F -superalgebra A with the trace (12) satisfies the $\mathbb{Z}/2\mathbb{Z}$ -graded identity with trace $\text{tr}(z)f = \sum_{i=1}^{t_{\bar{0}}} f|_{x_i=z x_i}$. Where $f(x_1, \dots, x_{t_1}, Y) \in F\langle X^{\mathbb{Z}_2} \rangle$ is any graded polynomial (without trace) of the type $(\dim_{\mathbb{Z}_2} A, \text{nd}(A) - 1, 1)$, alternating in the set $\{x_1, \dots, x_{t_0}\} \subseteq X_{\bar{0}}$. Here $t_{\bar{0}} = \dim B_{\bar{0}}$, $z \in X_{\bar{0}}$.*

Proof. A polynomial f of the type $(\dim_{\mathbb{Z}_2} A, \text{nd}(A) - 1, 1)$ is $(\dim_{\mathbb{Z}_2} A)$ -alternating in at least one set of graded variables. Suppose that $\{x_1, \dots, x_{t_0}\} \subseteq X_{\bar{0}}$ is the part of the even graded degree of this set. An elementary evaluation of variables of the set $\{z, x_1, \dots, x_{t_0}\}$ of the polynomial $g = \text{tr}(z)f - \sum_{i=1}^{t_0} f|_{x_i:=zx_i}$ must be semisimple. Moreover the set $\{x_1, \dots, x_{t_0}\}$ must be exchanged by pairwise different semisimple elements of the basis $D(2)$ of degree $\bar{0}$. Otherwise, we get zero.

Suppose that $z = b \in B_{\bar{0}}$ in our evaluation. Consider a polynomial $\tilde{f}(x_1, \dots, x_{t_0}, \dots)$ that is alternating in a homogeneous set of variables of even degree $\{x_1, \dots, x_{t_0}\} \subseteq X_{\bar{0}}$. It can be directly checked (see also [[5], Theorem J]) that for an arbitrary linear operator $\mathfrak{K} : B_{\bar{0}} \rightarrow B_{\bar{0}}$ and for all pairwise distinct basic elements b_1, \dots, b_{t_0} of $B_{\bar{0}}$ the equality $\text{Tr}(\mathfrak{K})\tilde{f}(b_1, \dots, b_{t_0}, \dots) = \tilde{f}(\mathfrak{K}(b_1), \dots, b_{t_0}, \dots) + \dots + \tilde{f}(b_1, \dots, \mathfrak{K}(b_{t_0}), \dots)$ holds in A , the replacement of other variables being arbitrary. Applying this observation to the linear operator $\mathfrak{T}(b)$ of the left multiplication by the element b we complete the proof. \square

Observe that in Lemma 25 it is enough to consider semisimple or radical evaluations of variables (not necessary elementary ones).

Lemma 26 *Let us take any $\beta \in \mathbb{N}_0^2$, $\gamma \in \mathbb{N}$. Consider a graded polynomial without trace $f(y_1, \dots, y_k) \in F\langle X^{\mathbb{Z}_2} \rangle$ of the type $(\beta; \gamma - 1; 1)$, and a pure trace polynomial $s(z_1, \dots, z_d) \in \mathcal{S}$ ($\{z_1, \dots, z_d\} \subseteq X^{\mathbb{Z}_2}$.) Then there exists a graded polynomial without trace $g_s(y_1, \dots, y_k, z_1, \dots, z_d) \in F\langle X^{\mathbb{Z}_2} \rangle$ such that $g_s \in \mathbb{Z}_2 T[f]$, and any finite dimensional superalgebra A with the parameters $\text{par}_{\mathbb{Z}_2}(A) = (\beta; \gamma)$ satisfies the graded identity with trace*

$$s(z_1, \dots, z_d) \cdot f(y_1, \dots, y_k) - g_s(y_1, \dots, y_k, z_1, \dots, z_d) = 0.$$

Proof. If $\text{par}_{\mathbb{Z}_2}(A) = (\beta; \gamma)$ then by Lemma 25 A satisfies the $\mathbb{Z}/2\mathbb{Z}$ -graded trace identity $\text{tr}(z)f(w_1x_1, w_2x_2, \dots, w_{t_0}x_{t_0}, Y) - (f(zw_1x_1, w_2x_2, \dots, w_{t_0}x_{t_0}, Y) + \dots + f(w_1x_1, w_2x_2, \dots, zw_{t_0}x_{t_0}, Y)) = 0$. Where $w_i \in (F\langle X^{\mathbb{Z}_2} \rangle^\#)_{\bar{0}}$ are arbitrary (possibly empty) graded monomials ($i = 1, \dots, t_0$), and $f(x_1, \dots, x_{t_0}, Y) \in F\langle X^{\mathbb{Z}_2} \rangle$ is any graded polynomial of the type $(\beta; \gamma - 1; 1)$, alternating in $\{x_1, \dots, x_{t_0}\} \subseteq X_{\bar{0}}$, $z \in X_{\bar{0}}$.

Let us take any graded monomials $u_i \in (F\langle X^{\mathbb{Z}_2} \rangle)_{\bar{0}}$ ($i = 1, \dots, n$), and any (possibly empty) graded monomials $v_j \in (F\langle X^{\mathbb{Z}_2} \rangle^\#)_{\bar{0}}$ ($j = 1, \dots, t_0$). Then by induction on the number $n \in \mathbb{N}_0$ of monomials u_i there exist a natural \tilde{n} , and graded monomials $\tilde{v}_{lj} \in (F\langle X^{\mathbb{Z}_2} \rangle^\#)_{\bar{0}}$ (possibly empty) such that $\text{tr}(u_1) \cdots \text{tr}(u_n) f(v_1x_1, \dots, v_{t_0}x_{t_0}, Y) = \sum_{l=1}^{\tilde{n}} f(\tilde{v}_{l1}x_1, \dots, \tilde{v}_{lt_0}x_{t_0}, Y) \pmod{\text{SIId}^{\mathbb{Z}_2}(A)}$. Moreover the monomials \tilde{v}_{lj} depend on the same variables as the monomial $u_1 \cdots u_n v_1 \cdots v_{t_0}$.

Hence for any pure trace polynomial $s(z_1, \dots, z_d) = \sum_{(j)} \alpha_{(j)} \text{tr}(u_{j_1}) \cdots \text{tr}(u_{j_n}) \in \mathcal{S}$ the algebra A satisfies the identity

$$s(z_1, \dots, z_d) \cdot f(x_1, \dots, x_{t_0}, Y) - g_s(x_1, \dots, x_{t_0}, Y, z_1, \dots, z_d) = 0.$$

Where $g_s \in \mathbb{Z}_2 T[f]$ is some graded polynomial that does not depend on A . Here $u_j \in (F\langle z_1, \dots, z_d \rangle)_{\bar{0}}$ are monomials, $\alpha_{(j)} \in F$. \square

Let A be a finite dimensional F -superalgebra with the semisimple part $B = B_{\bar{0}} \oplus B_{\bar{1}}$, and the Jacobson radical $J = J_{\bar{0}} \oplus J_{\bar{1}}$. Let us denote $\dim B_{\theta} = t_{\theta}$, $\dim J_{\theta} = q_{\theta}$ for any $\theta \in \mathbb{Z}/2\mathbb{Z}$.

Given a number $\nu \in \mathbb{N}$ take a set $\Lambda_{\nu} = \{\lambda_{\theta ij} | \theta \in \mathbb{Z}/2\mathbb{Z}, 1 \leq i \leq \nu, 1 \leq j \leq t_{\theta} + q_{\theta}\}$. Consider the free commutative associative unitary algebra $F[\Lambda_{\nu}]^{\#}$ generated by Λ_{ν} , and the associative algebra $\mathcal{P}_{\nu}(A) = F[\Lambda_{\nu}]^{\#} \otimes_F A$. The algebra $\mathcal{P}_{\nu}(A)$ has the structure of $F[\Lambda_{\nu}]^{\#}$ -module defined by $a \cdot f = f \cdot a = f \cdot (\sum_i f_i \otimes a_i) = \sum_i (f f_i) \otimes a_i$, for any $f, f_i \in F[\Lambda_{\nu}]^{\#}$, $a_i \in A$, $a = \sum_i f_i \otimes a_i \in \mathcal{P}_{\nu}(A)$.

The $\mathbb{Z}/2\mathbb{Z}$ -grading on $\mathcal{P}_{\nu}(A)$ is induced from A assuming that $F[\Lambda_{\nu}]^{\#}$ is trivially graded. Define an $F[\Lambda_{\nu}]^{\#}$ -linear map $\text{tr} : (\mathcal{P}_{\nu}(A))_{\bar{0}} \rightarrow F[\Lambda_{\nu}]^{\#}$ by the equalities

$$\text{tr}(a) = \text{tr}\left(\sum_{i=1}^{t_{\bar{0}}} f_i \otimes b_{i\bar{0}} + \sum_{j=1}^{q_{\bar{0}}} \tilde{f}_j \otimes r_{j\bar{0}}\right) = \sum_{i=1}^{t_{\bar{0}}} f_i \text{Tr}(\mathfrak{T}(b_{i\bar{0}})). \quad (13)$$

tr is well defined in $\mathcal{P}_{\nu}(A)$. Here $b_{i\bar{0}} \in B_{\bar{0}}$, $r_{je} \in (J(A))_{\bar{0}}$, $f_i, \tilde{f}_j \in F[\Lambda_{\nu}]^{\#}$. \mathfrak{T} is the left regular representation of $B_{\bar{0}}$, and Tr is the usual trace.

Lemma 27 $\mathcal{P}_{\nu}(A)$ satisfies the graded identity with trace $(\mathcal{X}_{t_{\bar{0}}}(x) \cdot x)^{\text{nd}(A)} = 0$. Here $\mathcal{X}_{t_{\bar{0}}}(x)$ is the Cayley-Hamilton polynomial of the degree $t_{\bar{0}}$, $x \in X_{\bar{0}}$.

Proof. The algebra $B_{\bar{0}}$ with the trace (12) satisfies the identity with trace $\mathcal{X}_{t_{\bar{0}}}(x) \cdot x = 0$ ([19], [20]). $F[\Lambda_{\nu}]^{\#}$ is commutative non-nilpotent algebra. $F[\Lambda_{\nu}]^{\#} \otimes_F (J(A))_{\bar{0}}$ is a nilpotent ideal of $(\mathcal{P}_{\nu}(A))_{\bar{0}}$ of the degree $\text{nd}(A)$. Hence the neutral component $(\mathcal{P}_{\nu}(A))_{\bar{0}} = (F[\Lambda_{\nu}]^{\#} \otimes_F B_{\bar{0}}) \oplus (F[\Lambda_{\nu}]^{\#} \otimes_F (J(A))_{\bar{0}})$ satisfies the full linearization of the identity $(\mathcal{X}_{t_{\bar{0}}}(x) \cdot x)^{\text{nd}(A)} = 0$. \square

Let $\{\hat{b}_{\theta 1}, \dots, \hat{b}_{\theta t_{\theta}}\}$ be a basis of the graded component B_{θ} ($\theta \in \mathbb{Z}/2\mathbb{Z}$) of the semisimple part B of A , $\dim B_{\theta} = t_{\theta}$. Also let $\{\hat{r}_{\theta 1}, \dots, \hat{r}_{\theta q_{\theta}}\}$ be a basis of the graded component J_{θ} of the Jacobson radical $J = J(A)$, $\dim J_{\theta} = q_{\theta}$. Recall that for a superalgebra with elementary decomposition all these bases may be chosen in the set $D \cup U$ ((2), (3), Lemma 1). Take elements

$$y_{\theta i} = \sum_{j=1}^{t_{\theta}} \lambda_{\theta ij} \otimes \hat{b}_{\theta j} + \sum_{j=1}^{q_{\theta}} \lambda_{\theta ij+t_{\theta}} \otimes \hat{r}_{\theta j} \in \mathcal{P}_{\nu}(A), \quad \theta \in \mathbb{Z}/2\mathbb{Z}, \quad 1 \leq i \leq \nu. \quad (14)$$

The elements $y_{\theta i}$ are $\mathbb{Z}/2\mathbb{Z}$ -homogeneous of degree θ ($\theta \in \mathbb{Z}/2\mathbb{Z}$; $1 \leq i \leq \nu$). Given a positive integer ν consider the F -subalgebra $\mathcal{F}_{\nu}(A) = \langle y_{\theta i} | \theta \in \mathbb{Z}/2\mathbb{Z}, 1 \leq i \leq \nu \rangle$ of $\mathcal{P}_{\nu}(A)$ generated by the set $Y_{\nu}^{\mathbb{Z}_2} = \{y_{\theta i} | \theta \in \mathbb{Z}/2\mathbb{Z}, 1 \leq i \leq \nu\}$. $\mathcal{F}_{\nu}(A)$ is $\mathbb{Z}/2\mathbb{Z}$ -graded.

Any map φ of generators to arbitrary homogeneous elements $\tilde{a}_{\theta i} \in A_{\theta}$ ($\tilde{a}_{\theta ij} \in F$)

$$\varphi : y_{\theta i} \mapsto \tilde{a}_{\theta i} = \sum_{j=1}^{t_{\theta}} \tilde{\alpha}_{\theta ij} \hat{b}_{\theta j} + \sum_{j=1}^{q_{\theta}} \tilde{\alpha}_{\theta ij+t_{\theta}} \hat{r}_{\theta j} \in A_{\theta} \quad (\theta \in \mathbb{Z}/2\mathbb{Z}, \quad i = 1, \dots, \nu) \quad (15)$$

can be extended to the graded homomorphism of F -superalgebras $\varphi : \mathcal{F}_{\nu}(A) \rightarrow A$, also inducing the graded homomorphism $\tilde{\varphi} : \mathcal{P}_{\nu}(A) \rightarrow A$ defined by the following

equalities

$$\tilde{\varphi}((\lambda_{\theta_1 i_1 j_1} \cdots \lambda_{\theta_k i_k j_k}) \otimes a) = (\tilde{\alpha}_{\theta_1 i_1 j_1} \cdots \tilde{\alpha}_{\theta_k i_k j_k}) \cdot a \quad \forall a \in A. \quad (16)$$

The homomorphism $\tilde{\varphi}$ preserves the trace defined by (13) on $\mathcal{P}_\nu(A)$ and by (12) on A .

Elements of $\mathcal{F}_\nu(A)$ are called quasi-polynomials in the variables $Y_\nu^{\mathbb{Z}_2}$. Products of the generators $y_{\theta_i} \in Y_\nu^{\mathbb{Z}_2}$ of the algebra $\mathcal{F}_\nu(A)$ are called quasi-monomials. We have also $\text{Id}^{\mathbb{Z}_2}(\mathcal{F}_\nu(A)) \supseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{P}_\nu(A)) = \text{Id}^{\mathbb{Z}_2}(A)$ for any $\nu \in \mathbb{N}$.

$\mathcal{F}_\nu(A)$ is a finitely generated PI-algebra. By Shirshov's height theorem [22] $\mathcal{F}_\nu(A)$ has a finite height and a finite Shirshov's basis. Shirshov's basis of an algebra always can be chosen in the set of monomials over the generators ([5], [22]). Thus we can suppose that a Shirshov's basis of $\mathcal{F}_\nu(A)$ consists of homogeneous in the grading elements of $\mathcal{F}_\nu(A)$. More precisely, there exist a positive integer \mathcal{H} , and homogeneous in the grading elements $w_1, \dots, w_d \in \mathcal{F}_\nu(A)$ such that any element $u \in \mathcal{F}_\nu(A)$ has the form $u = \sum_{(i)=(i_1, \dots, i_k)} \alpha_{(i)} w_{i_1}^{c_1} \cdots w_{i_k}^{c_k}$, where $k \leq \mathcal{H}$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$, $c_j \in \mathbb{N}$, $\alpha_{(i)} \in F$.

Consider the polynomials $\hat{s}_{il} = \text{tr}(w_i^{2l}) \in F[\Lambda_\nu]^\#$ ($i = 1, \dots, d$, $l = 1, \dots, t_0$). Then $\hat{F} = F[\hat{s}_{il} \mid 1 \leq i \leq d; 1 \leq l \leq t_0]^\#$ is the associative commutative non-graded F -subalgebra of $F[\Lambda_\nu]^\#$ with the unit generated by $\{\hat{s}_{il}\}$, and by the unit of $F[\Lambda_\nu]^\#$.

Take the $\mathbb{Z}/2\mathbb{Z}$ -graded \hat{F} -subalgebra $\mathcal{T}_\nu(A) = \hat{F}\mathcal{F}_\nu(A)$ of $\mathcal{P}_\nu(A)$. Then $\mathcal{F}_\nu(A)$ is a graded subalgebra of $\mathcal{T}_\nu(A)$. An arbitrary map of type (15) can be uniquely extended to a graded homomorphism from $\mathcal{T}_\nu(A)$ to A preserving the traces (it is the restriction of $\tilde{\varphi}$ defined by (16) onto $\mathcal{T}_\nu(A)$).

Since for any $i = 1, \dots, d$ we have $w_i^2 \in (\mathcal{P}_\nu(A))_{\bar{0}}$ then it follows from Lemma 27 that all elements w_i^2 are algebraic of degree $\text{nd}(A)(t_0 + 1)$ over \hat{F} . Therefore, by Shirshov's height theorem, $\mathcal{T}_\nu(A)$ is finitely generated \hat{F} -module, where \hat{F} is Noetherian. By theorem of Beidar [4] the algebra $\mathcal{T}_\nu(A)$ is representable.

Let $V \subseteq F\langle X^{\mathbb{Z}_2} \rangle$ be a set of graded polynomials. We denote by $V(\mathcal{T}_\nu(A)) \leq \mathcal{T}_\nu(A)$ the verbal ideal generated by results of all appropriate substitutions of $\mathbb{Z}/2\mathbb{Z}$ -homogeneous elements of $\mathcal{T}_\nu(A)$ to any graded polynomial from V .

Lemma 28 *Given a set $V \subseteq F\langle X^{\mathbb{Z}_2} \rangle$, and any $\nu \in \mathbb{N}$ the verbal ideal*

$$\begin{aligned} V(\mathcal{T}_\nu(A)) = \text{Span}_F \{ & \mathfrak{s} \cdot v_1 \tilde{f}(u_1, \dots, u_n) v_2 \mid \mathfrak{s} \in \hat{F}; \quad u_i \in \mathcal{F}_\nu(A) \text{ are} \\ & \text{quasi-monomials; } \quad v_1, v_2 \in \mathcal{F}_\nu(A)^\# \text{ are quasi-monomials, possibly empty;} \\ & \tilde{f} \text{ is the full linearization of a multihomogeneous component of any } f \in V \} \end{aligned}$$

is a graded \hat{F} -closed ideal of $\mathcal{T}_\nu(A)$. The quotient algebra $\overline{\mathcal{T}}_\nu(A, V) = \mathcal{T}_\nu(A)/V(\mathcal{T}_\nu(A))$ is a representable \hat{F} -superalgebra. The ideal of graded identities of $\overline{\mathcal{T}}_\nu(A, V)$ satisfies $\text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V)) \supseteq \tilde{\Gamma} + \text{Id}^{\mathbb{Z}_2}(F\langle X_\nu^{\mathbb{Z}_2} \rangle / (\tilde{\Gamma} \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle))$, where $\tilde{\Gamma} = \text{Id}^{\mathbb{Z}_2}(A) + \mathbb{Z}_2 T[V]$.

Proof. It is clear that in case of characteristic zero any verbal ideal of an algebra can be generated by the results of all appropriate substitutions to the full linearizations

of multihomogeneous components of polynomials from the given set. Particularly, $V(\mathcal{T}_\nu(A))$ is generated by $\mathbb{Z}/2\mathbb{Z}$ -homogeneous elements, and hence, this is a graded ideal.

Any homogeneous element of $\mathcal{T}_\nu(A)$ has the form $c = \sum_i \mathfrak{s}_i \cdot u_i$, where $\mathfrak{s}_i \in \widehat{F}$, $u_i \in \mathcal{F}_\nu(A)$ are quasi-monomials, $\deg_{\mathbb{Z}/2\mathbb{Z}} u_i = \deg_{\mathbb{Z}/2\mathbb{Z}} c$ for all i . If \tilde{f} is a multilinear graded polynomial then $\tilde{f}(c_1, \dots, c_n) = \tilde{f}(\sum_{i_1} \mathfrak{s}_{1i_1} \cdot u_{1i_1}, \dots, \sum_{i_n} \mathfrak{s}_{ni_n} \cdot u_{ni_n}) = \sum_{(i_1, \dots, i_n)} (\prod_{l=1}^n \mathfrak{s}_{li_l}) \tilde{f}(u_{1i_1}, \dots, u_{ni_n})$ for any homogeneous elements $c_1, \dots, c_n \in \mathcal{T}_\nu(A)$. Here $u_{li_l} \in \mathcal{F}_\nu(A)$ are quasi-monomials of appropriate graded degrees, $\mathfrak{s}_{li_l} \in \widehat{F}$. We have also $\mathfrak{s} \cdot \tilde{f}(c_1, \dots, c_n) = \tilde{f}(\mathfrak{s} \cdot c_1, \dots, c_n) = \tilde{f}(c'_1, \dots, c_n) \in V(\mathcal{T}_\nu(A))$ for any $\mathfrak{s} \in \widehat{F}$.

Then the quotient algebra $\overline{\mathcal{T}}_\nu(A, V)$ is $\mathbb{Z}/2\mathbb{Z}$ -graded is an \widehat{F} -module. $\overline{\mathcal{T}}_\nu(A, V)$ is finitely generated over \widehat{F} , as well as $\mathcal{T}_\nu(A)$, and is also representable. $\mathcal{T}_\nu(A)$ is a graded F -subalgebra of $\mathcal{P}_\nu(A)$. Therefore we have $\text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V)) \supseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{T}_\nu(A)) \supseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{P}_\nu(A)) = \text{Id}^{\mathbb{Z}_2}(A)$. It is clear also that $V \subseteq \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$, hence $\Gamma = \text{Id}^{\mathbb{Z}_2}(A) + \mathbb{Z}_2 T[V] \subseteq \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$.

Let us denote the \mathbb{Z}_2 -T-ideals $\Gamma_1 = \text{Id}^{\mathbb{Z}_2}(F\langle X_\nu^{\mathbb{Z}_2} \rangle / (\tilde{\Gamma} \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle))$, and $\Gamma_2 = \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$. From the arguments above we have $\Gamma_1(F\langle X_\nu^{\mathbb{Z}_2} \rangle) \subseteq \tilde{\Gamma} \subseteq \Gamma_2$. Then for a multilinear graded polynomial $f(x_1, \dots, x_n) \in \Gamma_1$, for any homogeneous elements $u_i \in \mathcal{F}_\nu(A)$ ($\deg_G u_i = \deg_G x_i$), and for any $\mathfrak{s}_i \in \widehat{F}$ we obtain

$$f(\mathfrak{s}_1 u_1, \dots, \mathfrak{s}_n u_n) = \mathfrak{s}_1 \cdots \mathfrak{s}_n f(u_1, \dots, u_n) \in \widehat{F} \Gamma_1(\mathcal{F}_\nu(A)) \subseteq \widehat{F} \Gamma_2(\mathcal{F}_\nu(A)) \subseteq V(\mathcal{T}_\nu(A)),$$

since $\text{grk}(\mathcal{F}_\nu(A)) = \nu$. Therefore $f \in \Gamma_2$, and $\Gamma_1 \subseteq \Gamma_2$. \square

Definition 16 We say that a subset $V \subseteq F\langle X^{\mathbb{Z}_2} \rangle$ is multihomogeneous if for any $f \in V$ V contains all multihomogeneous components of f .

Lemma 29 Let $A = A_1 \times \dots \times A_\rho$ be the direct product of arbitrary finite dimensional superalgebras A_1, \dots, A_ρ . Suppose that $V \subseteq F\langle X^{\mathbb{Z}_2} \rangle$ is a multihomogeneous set, and ν is any positive integer. Let us take any $f(z_1, \dots, z_n) \in \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$, and any homogeneous polynomials $h_1, \dots, h_n \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ with $\deg_{\mathbb{Z}_2} h_l = \deg_{\mathbb{Z}_2} z_l$. Then the equality $f(h_1, \dots, h_n) = \sum_j \mathfrak{s}_j \cdot v_{j1} \tilde{f}_j(u_{j1}, \dots, u_{jn}) v_{j2} \pmod{\text{SID}^{\mathbb{Z}_2}(A_i)}$ holds for any $i = 1, \dots, \rho$. Here \tilde{f}_j are the full linearizations of some polynomials $f_j \in V$; $u_{jl}, v_{jl} \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ are monomials, v_{jl} may be empty; and $\mathfrak{s}_j \in \mathcal{S}$ are pure trace graded polynomials in the variables $X_\nu^{\mathbb{Z}_2}$.

Proof. Given a graded polynomial $f(z_1, \dots, z_n) \in \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$, and arbitrary homogeneous polynomials $h_1, \dots, h_n \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ of degrees according to variables of f , we have $f(\tilde{h}_1, \dots, \tilde{h}_n) \in V(\mathcal{T}_\nu(A))$. Here the quasi-polynomial $\tilde{h}_i = h_i(y_1, \dots, y_{2\nu})$ is obtained by replacement of the variables $x_l \in X_\nu^{\mathbb{Z}_2}$ by the corresponding elements $y_l \in Y_\nu^{\mathbb{Z}_2}$ (14). Since V is a multihomogeneous set then by Lemma 28 in the algebra $\mathcal{T}_\nu(A)$ we obtain the equality $f(\tilde{h}_1, \dots, \tilde{h}_n) = \sum_j \tilde{\mathfrak{s}}_j \cdot \tilde{v}_{j1} \tilde{f}_j(\tilde{u}_{j1}, \dots, \tilde{u}_{jn}) \tilde{v}_{j2}$, where \tilde{f}_j are the full linearizations of polynomials $f_j \in V$, $\tilde{u}_{jl} = u_{jl}(y_1, \dots, y_{m\nu}) \in \mathcal{F}_\nu(A)$ are quasi-monomials, $\tilde{\mathfrak{s}}_j = \mathfrak{s}_j(y_1, \dots, y_{m\nu}) \in \widehat{F}$ are pure trace quasi-polynomials, $\tilde{v}_{jl} = v_{jl}(y_1, \dots, y_{m\nu}) \in \mathcal{F}_\nu(A)$ are also quasi-monomials, possibly empty.

An arbitrary graded map $\varphi : Y_\nu^{\mathbb{Z}_2} \rightarrow A_i$ from the generating set (14) of $\mathcal{F}_\nu(A)$ into any subalgebra $A_i \subseteq A$ ($i = 1, \dots, \rho$) can be extended to the graded homomorphism $\tilde{\varphi} : \mathcal{T}_\nu(A) \rightarrow A_i$ preserving the trace. Thus the equality

$$\begin{aligned} f(h_1(a_1, \dots, a_{2\nu}), \dots, h_n(a_1, \dots, a_{2\nu})) - \sum_j \mathfrak{s}_j(a_1, \dots, a_{2\nu}) \cdot v_{j1}(a_1, \dots, a_{2\nu}) \times \\ \tilde{f}_j(u_{j1}(a_1, \dots, a_{2\nu}), \dots, u_{jn}(a_1, \dots, a_{2\nu})) v_{j2}(a_1, \dots, a_{2\nu}) = 0 \end{aligned}$$

holds in A_i for any elements $a_1, \dots, a_{2\nu} \in A_i$ of appropriate graded degrees. Therefore

$$\begin{aligned} f(h_1(x_1, \dots, x_{2\nu}), \dots, h_n(x_1, \dots, x_{2\nu})) - \sum_j \mathfrak{s}_j(x_1, \dots, x_{2\nu}) \cdot v_{j1}(x_1, \dots, x_{2\nu}) \times \\ \tilde{f}_j(u_{j1}(x_1, \dots, x_{2\nu}), \dots, u_{jn}(x_1, \dots, x_{2\nu})) v_{j2}(x_1, \dots, x_{2\nu}) = 0 \end{aligned}$$

is a graded identity with trace of the algebra A_i , for any $i = 1, \dots, \rho$. \square

Lemma 30 *Suppose that A_1, \dots, A_ρ are any \mathbb{Z}_2 PI-reduced algebras with $\text{ind}_{\mathbb{Z}_2}(A_i) = \kappa$ for all $i = 1, \dots, \rho$. Given a subset $V \subseteq \bigcup_{i=1}^\rho S_\mu(A_i)$ (for any $\mu \geq 1$), and a positive integer ν , there exists an F -finite dimensional superalgebra C_ν such that $\text{Id}^{\mathbb{Z}_2}(C_\nu) = \text{Id}^{\mathbb{Z}_2}\left(F\langle X_\nu^{\mathbb{Z}_2} \rangle / ((\bigcap_{i=1}^\rho \text{Id}^{\mathbb{Z}_2}(A_i) + \mathbb{Z}_2 T[V]) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle)\right)$.*

Proof. Let us take $A = A_1 \times \dots \times A_\rho$, and any $\nu \in \mathbb{N}$. By Lemma 11 we have $\bigcup_{i=1}^\rho S_\mu(A_i) = S_\mu(A)$ for any μ , hence $V \subseteq S_\mu(A)$. Observe that a set of boundary polynomials is always multihomogeneous. Then for any $f(z_1, \dots, z_n) \in \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_\nu(A, V))$, and any homogeneous polynomials $h_1, \dots, h_n \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ of appropriate graded degrees by Lemma 29 we have

$$f(h_1, \dots, h_n) = \sum_j \mathfrak{s}_j v_{j1} \tilde{f}_j(u_{j1}, \dots, u_{jn}) v_{j2} \pmod{\text{SI}^{\mathbb{Z}_2}(A_i)} \quad \forall i = 1, \dots, \rho.$$

Where $\tilde{f}_j \in \mathbb{Z}_2 T[V]$ are multilinear graded polynomials; $u_{jl}, v_{jl} \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ are monomials, v_{jl} may be empty; $\mathfrak{s}_j \in \mathcal{S}$, \mathfrak{s}_j depends on $X_\nu^{\mathbb{Z}_2}$.

A_i is a \mathbb{Z}_2 PI-reduced superalgebra, hence from Lemmas 21, 9 it follows that $\text{par}_G(A_i) = \text{ind}_G(A_i) = \text{ind}_G(A) = \kappa = (\beta; \gamma)$ ($\forall i = 1, \dots, \rho$). Any polynomial \tilde{f}_j has the type $(\beta; \gamma - 1; \mu)$ (as the multilinearization of the polynomial $f_j \in V \subseteq S_\mu(A)$). Then by Lemma 26 there exists a graded traceless polynomial $\tilde{g}_j \in \mathbb{Z}_2 T[\tilde{f}_j] \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle \subseteq \mathbb{Z}_2 T[V] \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle$ such that $\mathfrak{s}_j \tilde{f}_j(v_{j1}, \dots, v_{jn}) = \tilde{g}_j \pmod{\text{SI}^{\mathbb{Z}_2}(A_i)}$ for any $i = 1, \dots, \rho$. Therefore

$$f(h_1, \dots, h_n) = \sum_j v_{j1} \tilde{g}_j v_{j2} \pmod{\text{SI}^{\mathbb{Z}_2}(A_i)} \quad \forall i = 1, \dots, \rho,$$

where $g = \sum_j v_{j1} \tilde{g}_j v_{j2} \in \mathbb{Z}_2 T[V] \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle$. Hence we obtain that $f(h_1, \dots, h_n) - g \in \text{SI}^{\mathbb{Z}_2}(A_i) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle = \text{Id}^{\mathbb{Z}_2}(A_i) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle$ for all $i = 1, \dots, \rho$. Thus $f(h_1, \dots, h_n) \in (\text{Id}^{\mathbb{Z}_2}(A) + \mathbb{Z}_2 T[V]) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle$, and $f \in \text{Id}^{\mathbb{Z}_2}\left(F\langle X_\nu^{\mathbb{Z}_2} \rangle / ((\text{Id}^{\mathbb{Z}_2}(A) + \mathbb{Z}_2 T[V]) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle)\right)$.

By Lemma 28 we have also that $\text{Id}^{\mathbb{Z}_2}\left(F\langle X_{\nu}^{\mathbb{Z}_2}\rangle/((\text{Id}^{\mathbb{Z}_2}(A)+\mathbb{Z}_2T[V])\cap F\langle X_{\nu}^{\mathbb{Z}_2}\rangle)\right)\subseteq \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_{\nu}(A,V))$. The superalgebra $\overline{\mathcal{T}}_{\nu}(A,V)$ is representable. Hence by Lemma 3 there exists an F -finite dimensional superalgebra C_{ν} such that

$$\text{Id}^{\mathbb{Z}_2}(C_{\nu}) = \text{Id}^{\mathbb{Z}_2}(\overline{\mathcal{T}}_{\nu}(A,V)) = \text{Id}^{\mathbb{Z}_2}\left(F\langle X_{\nu}^{\mathbb{Z}_2}\rangle/((\text{Id}^{\mathbb{Z}_2}(A)+\mathbb{Z}_2T[V])\cap F\langle X_{\nu}^{\mathbb{Z}_2}\rangle)\right).$$

□

7 Graded identities of finitely generated PI-algebras.

Lemma 31 *Let F be a field of characteristic 0. Let Γ be a non-trivial ideal of graded identities of a finitely generated associative PI-superalgebra over F . Then there exists a finite dimensional associative F -superalgebra \tilde{A} satisfying the conditions $\text{Id}^{\mathbb{Z}_2}(\tilde{A})\subseteq \Gamma$, $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \text{ind}_{\mathbb{Z}_2}(\tilde{A})$, and $S_{\hat{\mu}}(\mathcal{O}(\tilde{A}))\cap \Gamma = \emptyset$ for some $\hat{\mu} \in \mathbb{N}_0$.*

Proof. Γ contains a non-trivial T-ideal $\tilde{\Gamma}$ of ordinary non-graded identities of a finitely generated associative PI-algebra. By [17] $\tilde{\Gamma}$ also contains the T-ideal of some finite dimensional non-graded algebra C^{od} . It is clear that $A = C^{\text{od}} \otimes_F F[\mathbb{Z}/2\mathbb{Z}]$ is a finite dimensional superalgebra with a $\mathbb{Z}/2\mathbb{Z}$ -grading induced by the natural grading of the group algebra $F[\mathbb{Z}/2\mathbb{Z}]$, and $\text{Id}^{\mathbb{Z}_2}(A) = \mathbb{Z}_2T[\text{Id}(C^{\text{od}})]$. Thus we have that Γ contains the ideal of graded identities $\text{Id}^{\mathbb{Z}_2}(A)$ of some finite dimensional $\mathbb{Z}/2\mathbb{Z}$ -graded algebra A . We assume by Lemma 16 that $A = \mathcal{O}(A) \times \mathcal{Y}(A)$ with the senior components A_1, \dots, A_{ρ} . It is clear that $\kappa = \text{ind}_{\mathbb{Z}_2}(\Gamma) \leq \kappa_1 = \text{ind}_{\mathbb{Z}_2}(A)$ (Lemma 6). If $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(A_i)$ for some $i = 1, \dots, \rho$ then $\kappa_1 = \text{ind}_{\mathbb{Z}_2}(A_i) = \kappa$. Thus, the case $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{O}(A))$ is trivial.

Assume that $\Gamma \not\subseteq \text{Id}^{\mathbb{Z}_2}(A_i)$ for all $i = 1, \dots, \rho_1$ ($1 \leq \rho_1 \leq \rho$), and $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(A'')$, where $A'' = \times_{i=\rho_1+1}^{\rho} A_i$, $A' = \times_{i=1}^{\rho_1} A_i$. Consider the set $V = S_{\tilde{\mu}}(A') \cap \Gamma$ for $\tilde{\mu} = \hat{\mu}(\mathcal{O}(A))$ (Definition 11). Take $\nu = \text{grk}(D)$ for a finitely generated PI-superalgebra D such that $\Gamma = \text{Id}^{\mathbb{Z}_2}(D)$. By Lemma 30 there exists a finite dimensional over F superalgebra C_{ν} such that $\text{Id}^{\mathbb{Z}_2}(C_{\nu}) = \text{Id}^{\mathbb{Z}_2}(F\langle X_{\nu}^{\mathbb{Z}_2}\rangle/((\text{Id}^{\mathbb{Z}_2}(A')+\mathbb{Z}_2T[V])\cap F\langle X_{\nu}^{\mathbb{Z}_2}\rangle))$.

Then $\tilde{A} = C_{\nu} \times A'' \times \mathcal{Y}(A)$ is a finite dimensional superalgebra. For any $f(x_1, \dots, x_n) \in \text{Id}^{\mathbb{Z}_2}(\tilde{A})$, and for any homogenous polynomials $h_1, \dots, h_n \in F\langle X_{\nu}^{\mathbb{Z}_2}\rangle$ of the appropriate graded degrees we have $f(h_1, \dots, h_n) = h + g$, where $h \in \text{Id}^{\mathbb{Z}_2}(A')$, $g \in \Gamma \cap \text{Id}^{\mathbb{Z}_2}(\mathcal{Y}(A))$. Therefore $h = f(h_1, \dots, h_n) - g \in \text{Id}^{\mathbb{Z}_2}(A') \cap \text{Id}^{\mathbb{Z}_2}(A'') \cap \text{Id}^{\mathbb{Z}_2}(\mathcal{Y}(A)) = \text{Id}^{\mathbb{Z}_2}(A) \subseteq \Gamma$. Hence $f(h_1, \dots, h_n) = h + g \in \Gamma$ for any $h_1, \dots, h_n \in F\langle X_{\nu}^{\mathbb{Z}_2}\rangle$, and by Remark 1 $\text{Id}^{\mathbb{Z}_2}(\tilde{A}) \subseteq \Gamma$. Particularly $\text{ind}_{\mathbb{Z}_2}(\tilde{A}) \geq \kappa$.

Suppose that $A'' \neq 0$. Then $\kappa_1 = \kappa = \text{ind}_{\mathbb{Z}_2}(\tilde{A})$. Thus either $\text{ind}_{\mathbb{Z}_2}(C_{\nu}) = \kappa$ implying $\mathcal{O}(\tilde{A}) = A'' \times \mathcal{O}(C_{\nu})$, or $\text{ind}_{\mathbb{Z}_2}(C_{\nu}) < \kappa$ implying $\mathcal{O}(\tilde{A}) = A''$. Since $S_{\mu}(A'') \cap \Gamma = \emptyset$ then in the first case we use Lemmas 11, 16, 10 and conclude for any $\mu \geq \max\{\hat{\mu}(C_{\nu}), \tilde{\mu}\}$ we have $S_{\mu}(\mathcal{O}(\tilde{A})) \cap \Gamma = S_{\mu}(C_{\nu}) \cap \Gamma \subseteq S_{\mu}(A') \cap \Gamma \subseteq V \subseteq \text{Id}^{\mathbb{Z}_2}(C_{\nu})$. Thus, in both cases $S_{\mu}(\mathcal{O}(\tilde{A})) \cap \Gamma = \emptyset$, and \tilde{A} satisfies the claims of the lemma.

If $A'' = 0$ then $\kappa \leq \text{ind}_{\mathbb{Z}_2}(\tilde{A}) = \max\{\text{ind}_{\mathbb{Z}_2}(C_{\nu}), \text{ind}_{\mathbb{Z}_2}(\mathcal{Y}(A))\} \leq \kappa_1$. If $\text{ind}_{\mathbb{Z}_2}(C_{\nu}) = \kappa_1 = \text{ind}_{\mathbb{Z}_2}(A')$ then $\mathcal{O}(\tilde{A}) = \mathcal{O}(C_{\nu})$, and by analogy with previous case we apply

Lemmas 16, 10 for any $\mu \geq \max\{\hat{\mu}(C_\nu), \tilde{\mu}\}$ and obtain $S_\mu(\mathcal{O}(\tilde{A})) \cap \Gamma = \emptyset$. It also follows from Lemma 10 that $\text{ind}_{\mathbb{Z}_2}(\tilde{A}) = \text{ind}_{\mathbb{Z}_2}(\Gamma)$, and \tilde{A} is also the desired algebra.

The last case $A'' = 0$, $\text{ind}_{\mathbb{Z}_2}(C_\nu) < \kappa_1$ gives $\tilde{A} = C_\nu \times \mathcal{Y}(A)$ with $\text{Id}^{\mathbb{Z}_2}(\tilde{A}) \subseteq \Gamma$, and $\kappa \leq \text{ind}_{\mathbb{Z}_2}(\tilde{A}) < \kappa_1 = \text{ind}_{\mathbb{Z}_2}(A)$. Then by the inductive step on $\text{ind}_{\mathbb{Z}_2}(A)$ we can assume that the assertion of the lemma holds in this case. \square

Lemma 31 with Lemma 10 implies the corollary.

Lemma 32 *Let F be a field of characteristic 0. Let Γ be a non-trivial ideal of graded identities of a finitely generated associative PI-superalgebra over F . Then Γ contains the ideal of graded identities of a finite dimensional associative F -superalgebra A satisfying $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \text{ind}_{\mathbb{Z}_2}(A)$, and $S_{\tilde{\mu}}(A) = S_{\tilde{\mu}}(\Gamma)$ for some $\tilde{\mu} \in \mathbb{N}_0$.*

Proof. Let us take a superalgebra A satisfying the claims of Lemma 31. Lemma 10 implies that $S_\mu(A) \supseteq S_\mu(\Gamma)$ for any $\mu \in \mathbb{N}_0$. On the other hand $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \text{ind}_{\mathbb{Z}_2}(A) = (\beta; \gamma)$. Consider an integer $\tilde{\mu} = \max\{\hat{\mu}(A), \hat{\mu}\}$, where $\hat{\mu}$ is defined by Lemma 31, and $\hat{\mu}(A)$ is taken from Definition 11. Then any polynomial $f \in S_{\tilde{\mu}}(A)$ has the type $(\beta; \gamma - 1; \tilde{\mu})$, and satisfies $f \notin \Gamma$ (since $f \in S_{\tilde{\mu}}(\mathcal{O}(A))$). Thus, $f \in S_{\tilde{\mu}}(\Gamma)$. \square

Theorem 1 *Let F be a field of characteristic zero. For any finitely generated associative $\mathbb{Z}/2\mathbb{Z}$ -graded PI-algebra D over F there exists a finite dimensional over F associative superalgebra \tilde{C} such that the \mathbb{Z}_2 - T -ideals of $\mathbb{Z}/2\mathbb{Z}$ -graded identities of D and \tilde{C} coincide.*

Proof. Let Γ be the ideal of graded identities of D . We use the induction on the Kemer index $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \kappa = (\beta; \gamma)$ of Γ .

Inductive basis. If $\text{ind}_{\mathbb{Z}_2}(\Gamma) = (0, 0; \gamma)$ then D is a nilpotent finitely generated F -superalgebra. Hence D is finite dimensional.

Inductive hypothesis. Lemma 31, and Lemma 32 imply $\Gamma \supseteq \text{Id}^{\mathbb{Z}_2}(A)$, where $A = \mathcal{O}(A) + \mathcal{Y}(A)$ is a finite dimensional superalgebra with $\text{ind}_{\mathbb{Z}_2}(\Gamma) = \text{ind}_{\mathbb{Z}_2}(A) = \kappa$. Moreover, $S_{\tilde{\mu}}(\Gamma) = S_{\tilde{\mu}}(\mathcal{O}(A)) = S_{\tilde{\mu}}(A) \subseteq \text{Id}^{\mathbb{Z}_2}(\mathcal{Y}(A))$ for some $\tilde{\mu} \in \mathbb{N}_0$.

Let A_1, \dots, A_ρ be the senior components of the algebra A . Denote $(t_0, t_1) = \beta(\Gamma) = \text{dims}_{\mathbb{Z}_2} A_i$, $t = t_0 + t_1$; $\gamma = \gamma(\Gamma) = \text{nd}(A_i)$ (for all $i = 1, \dots, \rho$). Let us take for any $i = 1, \dots, \rho$ the algebra $\tilde{A}_i = \mathcal{R}_{q_i, s}(A_i)$ defined by (4) for the senior component A_i with $q_i = \dim_F A_i$, $s = (t + 1)(\gamma + \tilde{\mu})$. \tilde{A}_i is a finite dimensional superalgebra. $\Gamma_i = \text{Id}^{\mathbb{Z}_2}(\tilde{A}_i) = \text{Id}^{\mathbb{Z}_2}(A_i)$, and $\text{dims}_{\mathbb{Z}_2} \tilde{A}_i = \text{dims}_{\mathbb{Z}_2} A_i = \beta$. The Jacobson radical $J(\tilde{A}_i) = (X_{q_i}^{\mathbb{Z}_2})/I$ of \tilde{A}_i is nilpotent of class at most $s = (t + 1)(\gamma + \tilde{\mu})$, where $I = \Gamma_i(B_i(X_{q_i}^{\mathbb{Z}_2})) + (X_{q_i}^{\mathbb{Z}_2})^s$. Here the algebra B_i can be considered as the semisimple part of A_i and of \tilde{A}_i simultaneously (Lemma 2, Lemma 21). Particularly, \tilde{A}_i are superalgebras with elementary decomposition. By Lemma 28, and Lemma 3 there exists a finite dimensional F -superalgebra C such that $\text{Id}^{\mathbb{Z}_2}(C) = \text{Id}^{\mathbb{Z}_2}(\tilde{\mathcal{T}}_\nu(\tilde{A}, \Gamma))$, where $\tilde{A} = \times_{i=1}^\rho \tilde{A}_i$, $\nu = \text{grk}(D)$.

Let us denote $\tilde{D}_\nu = F\langle X_\nu^{\mathbb{Z}_2} \rangle / ((\Gamma + K_{\tilde{\mu}}(\Gamma)) \cap F\langle X_\nu^{\mathbb{Z}_2} \rangle)$. Lemmas 6, 12 imply that $\text{ind}_{\mathbb{Z}_2}(\tilde{D}_\nu) \leq \text{ind}_{\mathbb{Z}_2}(\Gamma + K_{\tilde{\mu}}(\Gamma)) < \text{ind}_{\mathbb{Z}_2}(\Gamma)$. By the inductive step we obtain

$\text{Id}^{\mathbb{Z}_2}(\tilde{D}_\nu) = \text{Id}^{\mathbb{Z}_2}(\tilde{U})$ for a finite dimensional over F superalgebra \tilde{U} . Lemma 28 yields $\Gamma \subseteq \text{Id}^{\mathbb{Z}_2}(C \times \tilde{U})$.

Consider a multilinear polynomial $f(x_1, \dots, x_d) \in \text{Id}^{\mathbb{Z}_2}(C \times \tilde{U})$. Let us take any multihomogeneous polynomials $h_1, \dots, h_d \in F\langle X_\nu^{\mathbb{Z}_2} \rangle$ with $\deg_{\mathbb{Z}_2} h_i = \deg_{\mathbb{Z}_2} x_i$ ($i = 1, \dots, d$). We have $f(h_1, \dots, h_d) = g + h$ for some multihomogeneous graded polynomials $g \in \Gamma$, $h \in K_{\tilde{\mu}}(\Gamma)$. Then by Lemma 29 we obtain $h = f(h_1, \dots, h_d) - g \in \mathcal{ST} + \text{SIId}^{\mathbb{Z}_2}(\tilde{A}_i)$ for any $i = 1, \dots, \rho$. Hence $\tilde{h}(x_1, \dots, x_n) \in \mathcal{ST} + \text{SIId}^{\mathbb{Z}_2}(\tilde{A}_i)$ also holds for the multilinearization \tilde{h} of h . Lemma 23 implies that \tilde{h} is exact for A_i ($\tilde{h} \in K_{\tilde{\mu}}(\Gamma)$) for any $i = 1, \dots, \rho$.

Let us fix any $i = 1, \dots, \rho$. Assume that $\{c_1, \dots, c_{q_i}\}$ is a basis of A_i of homogeneous in the grading elements chosen in (2), (3) (Lemma 1), and fix the order of these basic elements. Suppose that $\tilde{a} = (r_1, \dots, r_{\tilde{s}}, b_{\tilde{s}+1}, \dots, b_n)$ is any elementary complete evaluation of \tilde{h} by elements of the algebra A_i , where $r_j \in J(A_i)$, $b_j \in B_i$, $\tilde{s} = \gamma - 1$. By Lemma 20 there exist a polynomial $\hat{h}_{\tilde{\mu}}(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{s}+\tilde{\mu}}, \tilde{X}, \tilde{Z}) \in \mathcal{ST} + \text{SIId}^{\mathbb{Z}_2}(\tilde{A}_i)$ of the type $(\beta, \gamma - 1, \tilde{\mu})$, and an elementary evaluation \tilde{u} in A_i such that $\hat{h}_{\tilde{\mu}}(\tilde{u}) = \tilde{h}(\tilde{a})$. Moreover, $\hat{h}_{\tilde{\mu}}$ is alternating in any \tilde{Y}_j ($j = 1, \dots, \tilde{s} + \tilde{\mu}$), and all variables from $\tilde{X} \cup \tilde{Z}$ are exchanged by semisimple elements. Then we have

$$\alpha_1 \hat{h}_{\tilde{\mu}} = \left(\prod_{d=1}^{\tilde{s}+\tilde{\mu}} \mathcal{A}_{\tilde{Y}_{d,0}} \mathcal{A}_{\tilde{Y}_{d,1}} \right) \hat{h}_{\tilde{\mu}} = \sum_j \left(\prod_{d=1}^{\tilde{s}+\tilde{\mu}} \mathcal{A}_{\tilde{Y}_{d,0}} \mathcal{A}_{\tilde{Y}_{d,1}} \right) (\tilde{s}_j \tilde{g}_j) \pmod{\text{SIId}^{\mathbb{Z}_2}(\tilde{A}_i)}, \quad (17)$$

where $\tilde{Y}_d = \tilde{Y}_{d,0} \cup \tilde{Y}_{d,1}$; $\alpha_1 \in F$, $\alpha_1 \neq 0$; $\tilde{g}_j \in \Gamma$, $\tilde{s}_j \in \mathcal{S}$. Denote $\{\zeta_1, \dots, \zeta_{\hat{n}}\}$ the variables $\tilde{Y} \cup \tilde{X} \cup \tilde{Z}$ of $\hat{h}_{\tilde{\mu}}$ (the first variables are from $\tilde{Y} = \bigcup_{d=1}^{\tilde{s}+\tilde{\mu}} \tilde{Y}_d$).

Given an element $u_l \in A_i$ of the mentioned above evaluation \tilde{u} of the polynomial $\hat{h}_{\tilde{\mu}}$ defined by Lemma 20 we take the element $\tilde{x}_{\pi(l)\theta_l} = x_{\pi(l)\theta_l} + I$ of the algebra \tilde{A}_i . Here $x_{\pi(l)\theta_l} \in X_{q_i}^{\mathbb{Z}_2}$ is a graded variable of graded degree $\theta_l = \deg_{\mathbb{Z}_2} u_l$, and $\pi(l)$ is the ordinal number of the element $u_l = c_{\pi(l)}$ in our basis of A_i , $1 \leq \pi(l) \leq q_i$ ($1 \leq l \leq \hat{n}$). Consider the following evaluation of $\hat{h}_{\tilde{\mu}}(\zeta_1, \dots, \zeta_{\hat{n}})$ in the algebra \tilde{A}_i

$$\begin{aligned} \zeta_l &= \varepsilon_{k_1} \tilde{x}_{\pi(l)\theta_l} \varepsilon_{k_2} \in J(\tilde{A}_i) \quad \text{if } \zeta_l \in \tilde{Y}, \quad u_l = c_{\pi(l)} \in \varepsilon_{k_1} A_i \varepsilon_{k_2}; \\ \zeta_l &= u_l \quad \text{if } \zeta_l \in \tilde{X} \cup \tilde{Z}. \end{aligned} \quad (18)$$

Suppose that in (17) the pure trace polynomial \tilde{s}_j depends essentially on \tilde{Y} . Then we get $\tilde{s}_j|_{(18)} = 0$, because the trace of a radical element is zero (12). If \tilde{s}_j does not depend on \tilde{Y} then $(\prod_{d=1}^{\tilde{s}+\tilde{\mu}} \mathcal{A}_{\tilde{Y}_{d,0}} \mathcal{A}_{\tilde{Y}_{d,1}})(\tilde{s}_j \tilde{g}_j) = \tilde{s}_j \tilde{g}_j$, where $\tilde{g}_j = ((\prod_{d=1}^{\tilde{s}+\tilde{\mu}} \mathcal{A}_{\tilde{Y}_{d,0}} \mathcal{A}_{\tilde{Y}_{d,1}}) \tilde{g}_j) \in \Gamma$. If $\tilde{g}_j|_{(18)} \neq 0$ in \tilde{A}_i then one of degree multihomogeneous components of \tilde{g}_j is a $\tilde{\mu}$ -boundary polynomial for \tilde{A}_i . And it is not a $\tilde{\mu}$ -boundary polynomial for Γ , because it belongs to Γ . This implies that $S_{\tilde{\mu}}(A) \neq S_{\tilde{\mu}}(\Gamma)$, which contradicts to the properties of A . Therefore, we have $\tilde{g}_j|_{(18)} = 0$.

Thus, in any case $\hat{h}_{\bar{\mu}}|_{(18)} = 0$ holds in the algebra \tilde{A}_i . Hence the evaluation

$$\begin{aligned} \zeta_l = v_l = \varepsilon_{k_1} x_{\pi(l)\theta_l} \varepsilon_{k_2} & \quad \text{if } \zeta_l \in \tilde{Y}, \quad u_l = c_{\pi(l)} \in \varepsilon_{k_1} A_i \varepsilon_{k_2}; \\ \zeta_l = v_l = u_l & \quad \text{if } \zeta_l \in \tilde{X} \bigcup \tilde{Z} \end{aligned}$$

of the polynomial $\hat{h}_{\bar{\mu}}$ in the algebra $B_i(X_{q_i}^{\mathbb{Z}_2})$ is equal to $\hat{h}_{\bar{\mu}}(v_1, \dots, v_{\hat{n}}) \in I = \Gamma_i(B_i(X_{q_i}^{\mathbb{Z}_2})) + (X_{q_i}^{\mathbb{Z}_2})^s$. Since $|\tilde{Y}| < s$, the polynomial $\hat{h}_{\bar{\mu}}$ is linear in \tilde{Y} , and the variables from $\tilde{X} \bigcup \tilde{Z}$ are replaced by semisimple elements, then we obtain $\hat{h}_{\bar{\mu}}(v_1, \dots, v_{\hat{n}}) \in \Gamma_i(B_i(X_{q_i}^{\mathbb{Z}_2}))$.

Consider the map $\varphi : x_{\pi(l)\theta_l} \mapsto c_{\pi(l)}$ ($l = 1, \dots, |\tilde{Y}|$), and $\varphi(b) = b$ for any $b \in B_i$. It is clear that φ can be extended to a graded homomorphism $\varphi : B_i(X_{q_i}^{\mathbb{Z}_2}) \rightarrow A_i$. Then $\varphi(\hat{h}_{\bar{\mu}}(v_1, \dots, v_{\hat{n}})) = \hat{h}_{\bar{\mu}}(\varphi(v_1), \dots, \varphi(v_{\hat{n}})) = \hat{h}_{\bar{\mu}}(\bar{u}) = \tilde{h}(\bar{a}) \in \varphi(\Gamma_i(B_i(X_{q_i}^{\mathbb{Z}_2}))) = \Gamma_i(A_i) = (0)$.

Therefore $\tilde{h}(\bar{a}) = 0$ holds in A_i for any elementary complete evaluation $\bar{a} \in A^n$ containing $\gamma - 1$ radical elements. Since \tilde{h} is a multihomogeneous exact polynomial for A_i , and $\gamma = \text{nd}(A_i)$ then $\tilde{h} \in \text{Id}^{\mathbb{Z}_2}(A_i)$. Hence $h \in \cap_{i=1}^{\rho} \text{Id}^{\mathbb{Z}_2}(A_i) = \text{Id}^{\mathbb{Z}_2}(\mathcal{O}(A))$, and $h \in \text{Id}^{\mathbb{Z}_2}(\mathcal{O}(A) \times \mathcal{Y}(A)) = \text{Id}^{\mathbb{Z}_2}(A) \subseteq \Gamma$. Then we have $f(h_1, \dots, h_d) = g + h \in \Gamma$ for any multihomogeneous polynomials $h_1, \dots, h_d \in F\langle X_{\nu}^{\mathbb{Z}_2} \rangle$ of appropriate graded degrees. The application of Remark 1 now implies that $\text{Id}^{\mathbb{Z}_2}(C \times \tilde{U}) \subseteq \Gamma$.

Therefore, $\Gamma = \text{Id}^{\mathbb{Z}_2}(C \times \tilde{U})$. Theorem is proved. \square

Observe that to be a PI-algebra is an essential condition for a finitely generated superalgebra D in Theorem 1, since a finite dimensional superalgebra is always a PI-algebra.

8 Specht problem.

Let $E = \langle e_i, i \in \mathbb{N} \mid e_i e_j = -e_j e_i, \forall i, j \rangle$ be the Grassmann algebra of infinite rank with the canonical $\mathbb{Z}/2\mathbb{Z}$ -grading $E = E_{\bar{0}} \oplus E_{\bar{1}}$ ($E_{\bar{0}}$ and $E_{\bar{1}}$ are the subspaces of E generated by all monomials in the generators of even and odd lengths respectively). Consider for a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ the Grassmann envelope $E(A) = A_{\bar{0}} \otimes E_{\bar{0}} \oplus A_{\bar{1}} \otimes E_{\bar{1}}$. Observe that the algebra $E(A)$ has the natural $\mathbb{Z}/2\mathbb{Z}$ -grading $E(A)_{\theta} = A_{\theta} \otimes E_{\theta}$, $\theta \in \mathbb{Z}/2\mathbb{Z}$. The next remark is obvious.

Remark 2 *If two PI-superalgebras A and B has the same $\mathbb{Z}/2\mathbb{Z}$ -graded identities then they have also the same ordinary non-graded polynomial identities. The T -ideal $\text{Id}(A)$ is the biggest T -ideal that is contained in $\text{Id}^{\mathbb{Z}_2}(A)$.*

Lemma 33 (Remark 3.7.6, [9]) *If two PI-superalgebras A and B have the same $\mathbb{Z}/2\mathbb{Z}$ -graded identities then the algebras $E(A)$, and $E(B)$ have the same ordinary non-graded polynomial identities.*

Proof. This is a simple consequence of the fact that the \mathbb{Z}_2 - T -ideals of superalgebras A and $E(A)$ related by some well understood invertible transformation (see, e.g.,

[12], [16], [9]). This transformation is completely algorithmic. Particularly, if two superalgebras are \mathbb{Z}_2 PI-equivalent then their Grassmann envelopes are also \mathbb{Z}_2 PI-equivalent. By Remark 2 the Grassmann envelopes of these superalgebras also have the same non-graded identities. \square

Theorem 2 *The T-ideal of polynomial identities of an associative PI-algebra over a field of characteristic zero coincides with the T-ideal of identities of the Grassmann envelope of some finitely generated PI-superalgebra.*

Proof. See Theorem 4.8.2 [9]. \square

Theorem 1 along with Theorem 2, and Lemma 33 implies the principle Kemer's classification theorem.

Theorem 3 *The T-ideal of polynomial identities of an associative PI-algebra over a field of characteristic zero coincides with the T-ideal of identities of the Grassmann envelope of some finite dimensional superalgebra.*

Theorem 3 immediately implies the positive solution of the Specht problem.

Theorem 4 (Theorem 2.4, [12]) *The T-ideal of polynomial identities of an associative PI-algebra over a field of characteristic zero is finitely generated as a T-ideal.*

Proof. Suppose that Γ is a T-ideal that is not finitely based. Then there exists an infinite sequence of multilinear polynomials $\{f_i(x_1, \dots, x_{n_i})\}_{i \in \mathbb{N}} \subseteq \Gamma$ satisfying the conditions $\deg f_i < \deg f_j$ for any $i < j$ and $f_i \notin T[f_1, \dots, f_{i-1}]$ for any $i \in \mathbb{N}$. Consider the T-ideals Γ_i generated by all consequences of the polynomial f_i of degrees strictly greater than $n_i = \deg f_i$ ($i = 1, 2, \dots$), and the T-ideal $\tilde{\Gamma} = \sum_{i \in \mathbb{N}} \Gamma_i$. Then we have $f_i \notin \tilde{\Gamma}$ for any $i \in \mathbb{N}$. By Theorem 3 we obtain $\tilde{\Gamma} = \text{Id}(E(C))$ for a finite dimensional superalgebra C .

Lemma 1 implies $E(C) = E(B) \oplus E(J)$, where B is the semisimple part of C , J is the Jacobson radical of C . Consider a polynomial f_k from our sequence, such that $\deg f_k = n_k > \text{nd}(C)$, and consider an evaluation in $E(C)$ of f_k of the type $x_i = a_i = c_{\delta_i} \otimes g_{\delta_i}$, where $c_{\delta_i} \in B_{\delta_i} \cup J_{\delta_i}$, $g_{\delta_i} \in E_{\delta_i}$, $\delta_i = \bar{0}, \bar{1}$ ($i = 1, \dots, n_k$). If for any $i = 1, \dots, n_k$ the elements c_{δ_i} are radical then $f_k(a_1, \dots, a_{n_k}) = 0$ in $E(C)$, since $n_k > \text{nd}(C)$. Suppose that $x_{\hat{l}} = b_{\delta_{\hat{l}}} \otimes g_{\delta_{\hat{l}}}$, where $b_{\delta_{\hat{l}}} \in B_{\delta_{\hat{l}}}$ for some \hat{l} . Then for any element $\tilde{g}_0 \in E_0$ we obtain $f_k(\dots, a_{\hat{l}}, \dots) \tilde{g}_0 = f_k(\dots, b_{\delta_{\hat{l}}} \otimes (g_{\delta_{\hat{l}}} \cdot \tilde{g}_0), \dots) = f_k(\dots, a_{\hat{l}} \cdot (1_B \otimes \tilde{g}_0), \dots) = 0$, since $f_k(x_1, \dots, x_{\hat{l}} \cdot x_0, \dots, x_{n_k}) \in \tilde{\Gamma}$, $x_0 \in X$, 1_B is the unit of B . It implies $f_k(a_1, \dots, a_{n_k}) = 0$. Therefore $f_k \in \tilde{\Gamma}$, that contradicts to the choice of $\tilde{\Gamma}$. \square

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